Symmetries and Differential Equations

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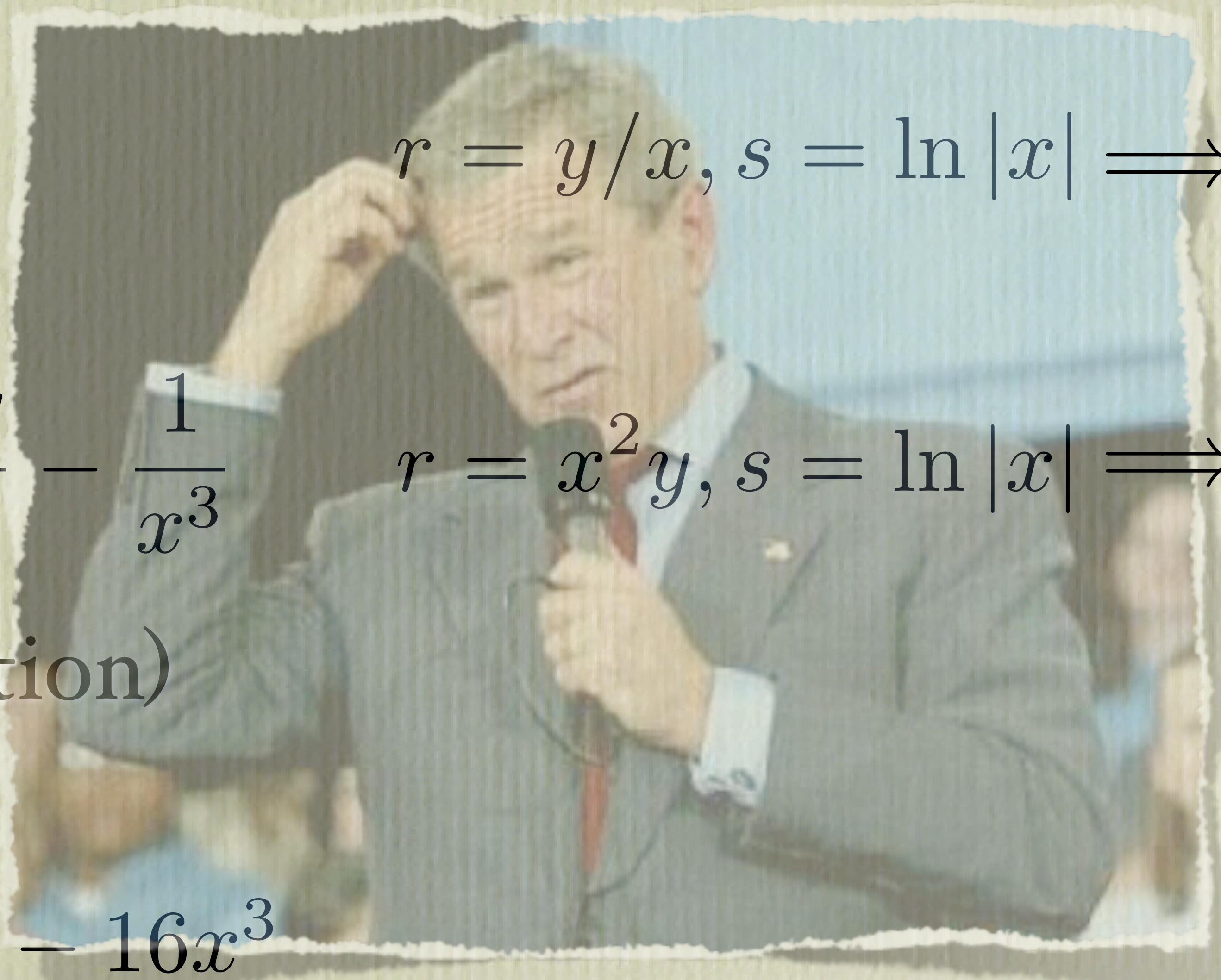
What will we do today?

- The why: Motivation
- The what: What are symmetries of ODEs
- The how: Solving ODEs using symmetries
- Another how: Finding symmetries
- Higher order ODEs + PDEs

Motivation

How would you solve...



$$\frac{dy}{dx} = F(\frac{y}{x}) \qquad r = y/x, s = \ln|x| \Longrightarrow \frac{ds}{dr} = \frac{1}{F(r) - r}$$

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3} \qquad r = x^2 y, s = \ln|x| \Longrightarrow \frac{ds}{dr} = \frac{1}{r^2 - 1}$$

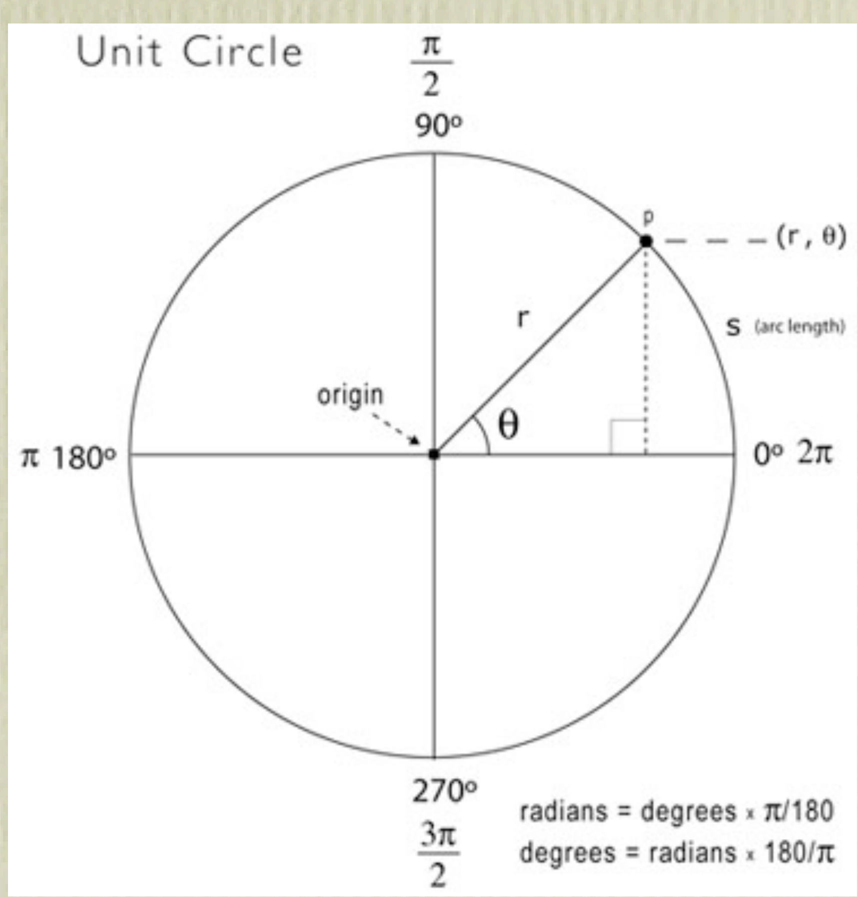
(Ricatti Equation)

$$\frac{dy}{dx} = \frac{y - 4xy^2 - 16x^3}{y^3 + 4x^2y + x}$$

What are symmetries?

Wikipedia: A symmetry of an object is a physical or mathematical feature of the object (observed or intrinsic) that is "preserved" under some transformation

Example: Circle



- Invariant w.r.t to rotations
- Invariant w.r.t to reflections

Symmetries of ODEs

Symmetry of an ODE =
transformation that map solutions to solutions...

For a first order ODE: $$\frac{dy}{dx} = \omega(x, y)$$

a symmetry is a transformation of the plane:

$$\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$$

such that:

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}) \quad \text{when} \quad \frac{dy}{dx} = \omega(x, y)$$

Lie Symmetries

For practical cases we will only treat one-parameter Lie groups (:= Lie symmetries). Namely,

$$\Gamma_\varepsilon : (x,y) \mapsto (\hat{x}(x,y;\varepsilon); \hat{y}(x,y;\varepsilon))$$

such that:

- Γ_ε is a symmetry about $\varepsilon = 0$
- $\Gamma_0 = Id$
- $\Gamma_\delta \Gamma_\varepsilon = \Gamma_{\delta+\varepsilon}$
- Γ_ε is analytic at $\varepsilon = 0$

Examples:

$$\Gamma : (x,y) \mapsto (x+\varepsilon, y)$$

$$\Gamma : (x,y) \mapsto (x, y+\varepsilon)$$

$$\Gamma : (x,y) \mapsto (x, \varepsilon y)$$

$$\Gamma : (x,y) \mapsto (x, e^\varepsilon y)$$

The Symmetry Condition

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y})$$

On solution curves $y = y(x)$, so $(\hat{x}, \hat{y})$ can be thought of as functions of x only: $(\hat{x}(x, y(x)), \hat{y}(x, y(x)))$

We can use the chain rule to write:

$$\omega(\hat{x}, \hat{y}) = \frac{D_x \hat{y}}{D_x \hat{x}} = \frac{\hat{y}_x + y' \hat{y}_y}{\hat{x}_x + y' \hat{x}_y}$$

Examples

But $y' = \omega(x, y)$

so we get the <u>symmetry condition</u>:

$$\omega(\hat{x}, \hat{y}) = \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y}$$

Finding Invariant Solutions

Since the Lie symmetry is analytic at $\varepsilon = 0$:

$$\hat{x} = x + \varepsilon\xi(x,y) + O(\varepsilon^2)$$

$$\hat{y} = y + \varepsilon\eta(x,y) + O(\varepsilon^2)$$

Yaron, draw a picture...

Sometimes there are invariant solutions, namely solutions that are mapped to themselves by the transformation.

Example

Solution $y(x)$ is invariant $\iff$ its tangent is parallel to $\mathbb{X}$

$$\iff (\xi(x,y), \eta(x,y)) \perp (y'(x), 1)$$

$$\iff \eta(x,y) - y'(x)\xi(x,y) = 0$$

$$\iff \bar{Q}(x,y) = 0$$

where the characteristic is $\bar{Q}(x,y) = \eta(x,y) - \omega(x,y)\xi(x,y)$

Solving ODEs using Symmetries

If $(x, y) \mapsto (x, y + \varepsilon)$ is a symmetry, solving the equation is easy...

<u>A. Newell (Spring 2009):</u> "Even the most stubborn equations can be solved if you find the right coordinates..."

<u>Goal</u>: Find new coordinates that have vertical symmetry!

$$(x, y) \mapsto (r(x, y), s(x, y))$$

Ah? Explanation + Examples

such that $$(\hat{r}, \hat{s}) = (r(\hat{x}, \hat{y}), s(\hat{x}, \hat{y})) = (r, s + \varepsilon)$$

(r, s) are called canonical coordinates, and are defined (not uniquely) by:

$$\mathbb{X}r = 0 \qquad \mathbb{X}s = 1$$

Notice: Canonical coordinates are not defined on invariant points.

Finding Symmetries

In general, the symmetry condition:

$$\omega(\hat{x},\hat{y}) = \frac{\hat{y}_x + \omega(x,y)\hat{y}_y}{\hat{x}_x + \omega(x,y)\hat{x}_y}$$ ⟵ DIFFICULTY

is a <u>nonlinear</u> PDE for $(\hat{x},\hat{y})$

So solving it to find symmetries might be more difficult...

<u>Einstein</u>: "In the middle of difficulty lies opportunity."

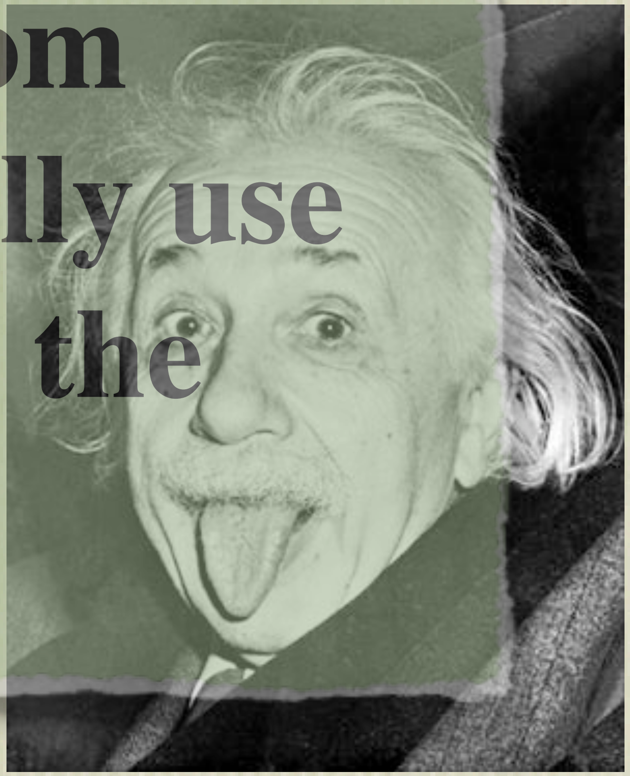
Expand the symmetry condition in ε and take the MIDDLE ε^1 term.

Linearized Symmetry Condition:

$$\bar{Q}_x + \omega(x,y)\bar{Q}_y = \omega_y(x,y)\bar{Q}$$

<u>Or instead...</u>

For ODEs derived from applications, we can usually use the applications to find the symmetries!



Conclusions and such...

- The method is similar for higher order ODEs : Any Lie symmetry we find can reduce the order of the equation by at least one.
- For PDEs, this is an active research topic, and there are many open problems.
- There are many other techniques for finding symmetries, but...
- Since finding symmetries can be difficult, people take a symmetry and classify all the equations that have that symmetry instead.

Thank you!