The Problem of Radiation-Reaction

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Outline

- Crash course in Electromagnetism
- Super crash course in Special Relativity
- The Problem of Radiation-Reaction
- The Lorentz-Abraham-Dirac Equation
- The Landau-Lifshitz Equation
- Study of the Landau-Lifshitz Equation

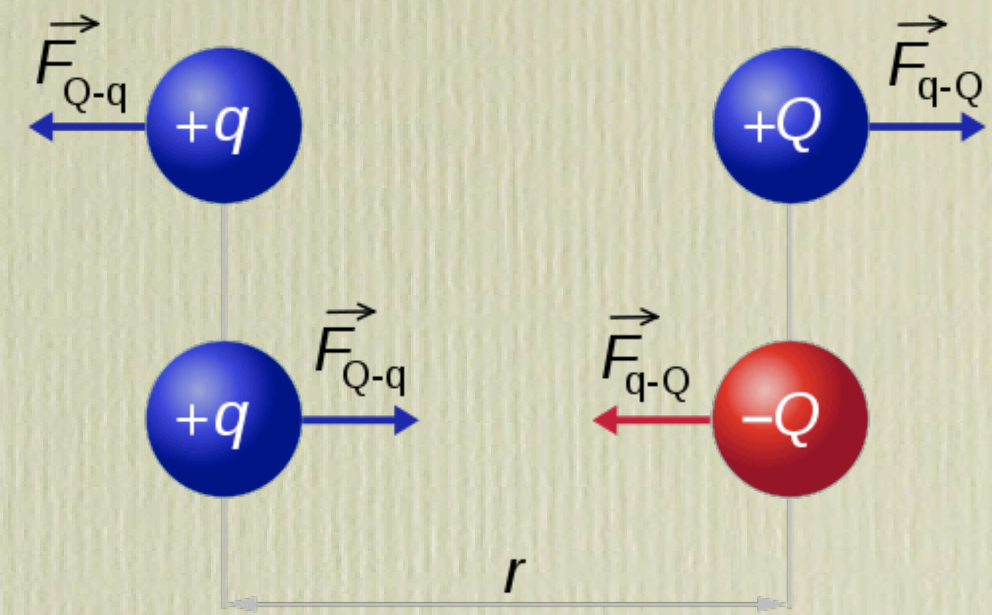
Coulomb's Law and the Electric Field

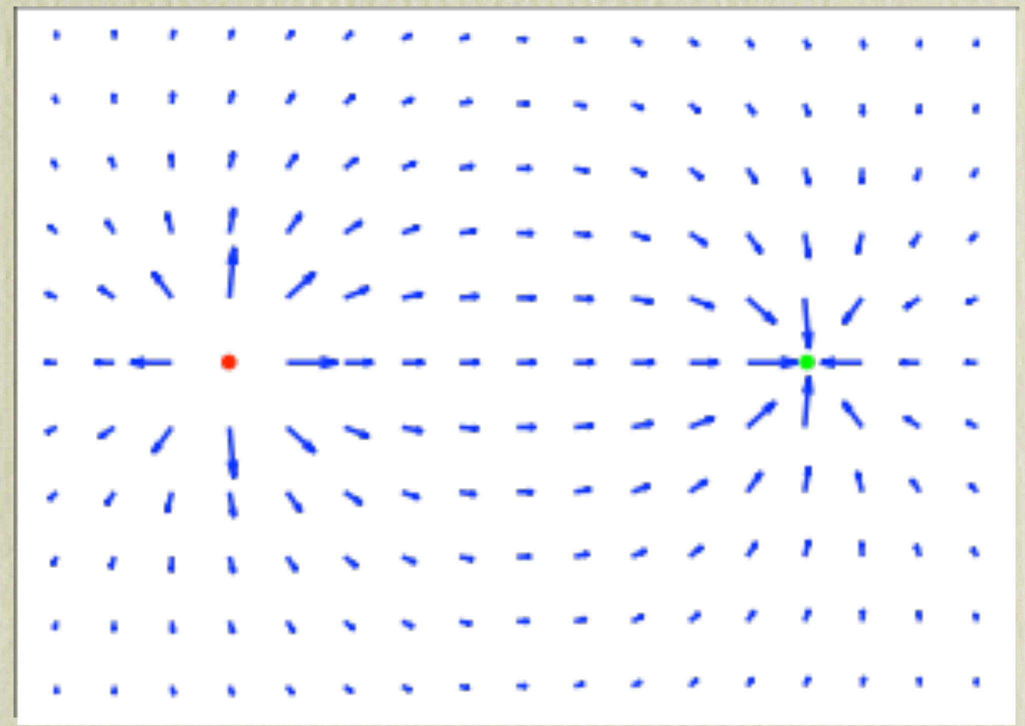
Newton's second law: $\vec{F} = m\ddot{\vec{x}}$

The force exerted by a charge Q at $\vec{x}_0$ on a charge q at $\vec{x}$ is the Coulomb's force:

$$\vec{F} = \underbrace{K\frac{qQ}{|\vec{x}-\vec{x}_0|^3}(\vec{x}-\vec{x}_0)}_{:= q\vec{E}(\vec{x})}$$

where K is the Coulomb's constant



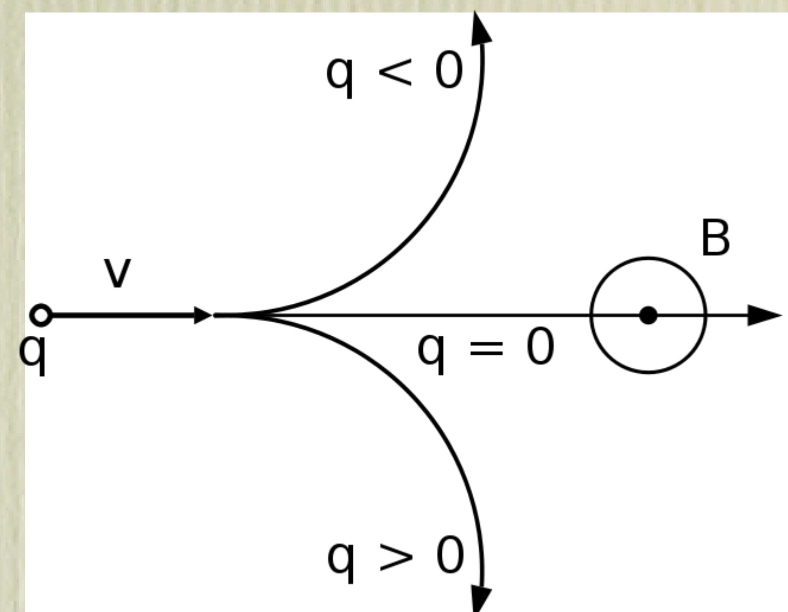


Coulomb's law only works for static charges

Lorentz Force Equation

When the charge q is not at rest and has velocity $\vec{v}$, the force has an additional component:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$



where $\vec{E}$ is the electric field, and $\vec{B}$ is called the magnetic field.

This is the Lorentz force.

Together with Newton's second law it gives the Lorentz force equation:

$$m\ddot{\vec{x}} = q(\vec{E} + \vec{v} \times \vec{B})$$

In order to fully describe a system of charges, we need to determine the electric and magnetic fields.

Maxwell's Equations

Gauss's la … rge density

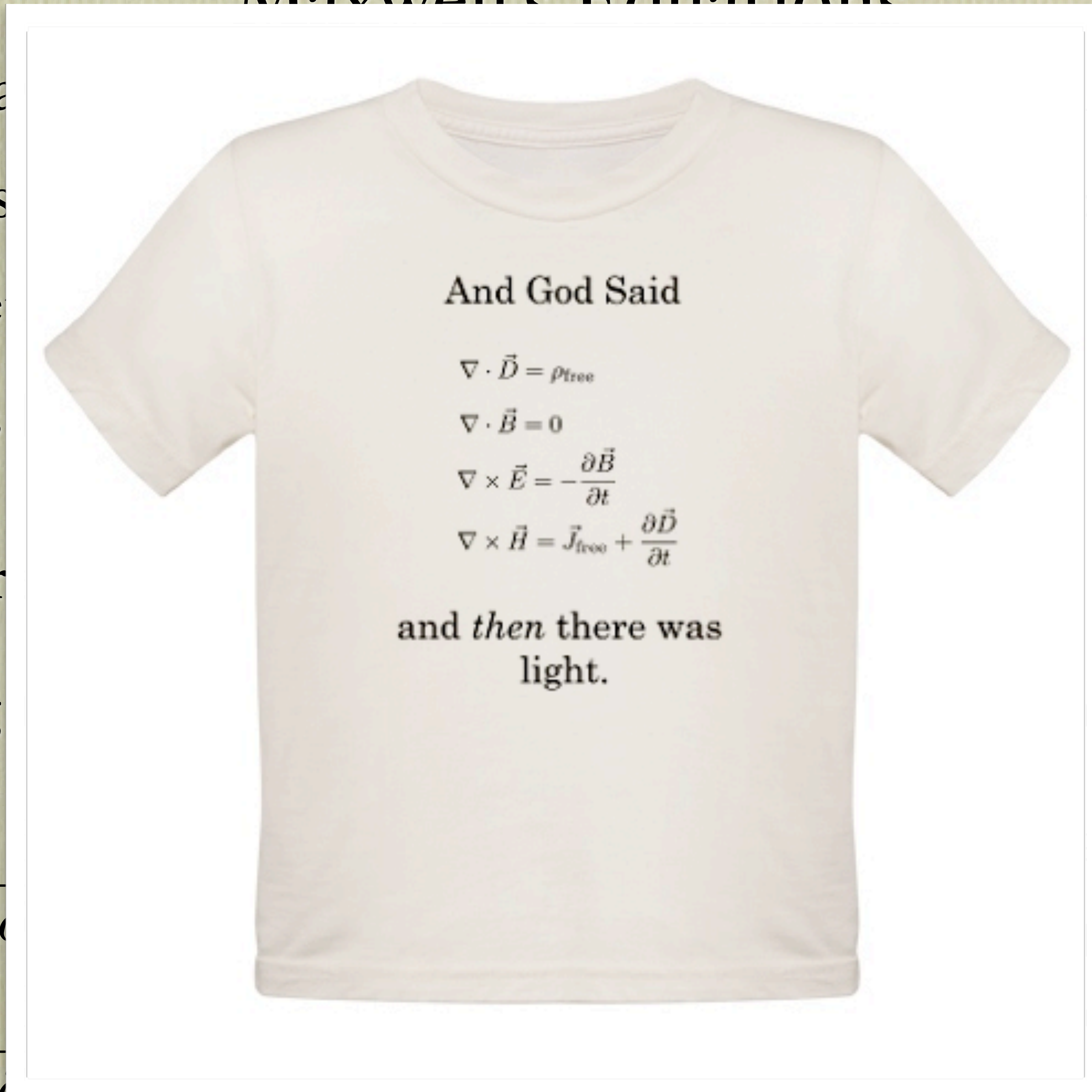
Ampère's … rent density

No magne … $\nabla \times \vec{A}$

Faraday's … $-\nabla\phi - \frac{\partial \vec{A}}{\partial t}$

$\vec{A}$ & ϕ ar

Plugging … gives:



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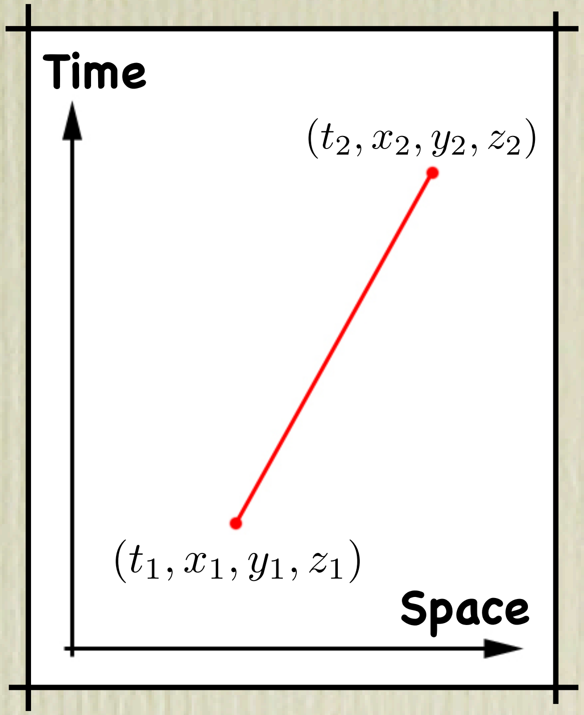
Optics $\subsetneq$ Electromagnetism

The Special Theory of Relativity (1905)

Postulate: The speed of light c is the same for all observers

Einstein: The electric and magnetic fields are the manifestation of the same field, viewed differently by different observers.



$$cdt = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Introduce the notation: $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$

and the matrix:

$$\eta_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and we can rewrite the condition as:

$$\eta_{\alpha\beta} dx^\alpha dx^\beta = 0$$

(Sum over repeated indices)

Mathematically, this turns spacetime into a 4D manifold with an 'inner-product': $\langle u, v \rangle = \eta_{\alpha\beta} u^\alpha v^\beta$

Covariant Formulation of Maxwell's equations

Write the non-homogenous Maxwell's equations in components:

$$\nabla \cdot \vec{E} = 4\pi\rho \qquad \Longrightarrow \qquad \frac{\partial E_x}{\partial x^1} + \frac{\partial E_y}{\partial x^2} + \frac{\partial E_z}{\partial x^3} = 4\pi\rho$$

$$\nabla \times \vec{B} - \frac{\partial \vec{E}}{\cdots} \cdots 4\pi \vec{J} \Longrightarrow -\frac{\partial E_x}{\cdots} + \frac{\partial B_z}{\cdots} - \frac{\partial B_y}{\cdots x^3} = 4\pi J_x$$

We can wr[illegible] $\cdots \beta$

Geometry intermission

with $F^{\alpha\beta} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$ and $J^\alpha = (\rho, \vec{J})$

how do we get the homogenous equations?

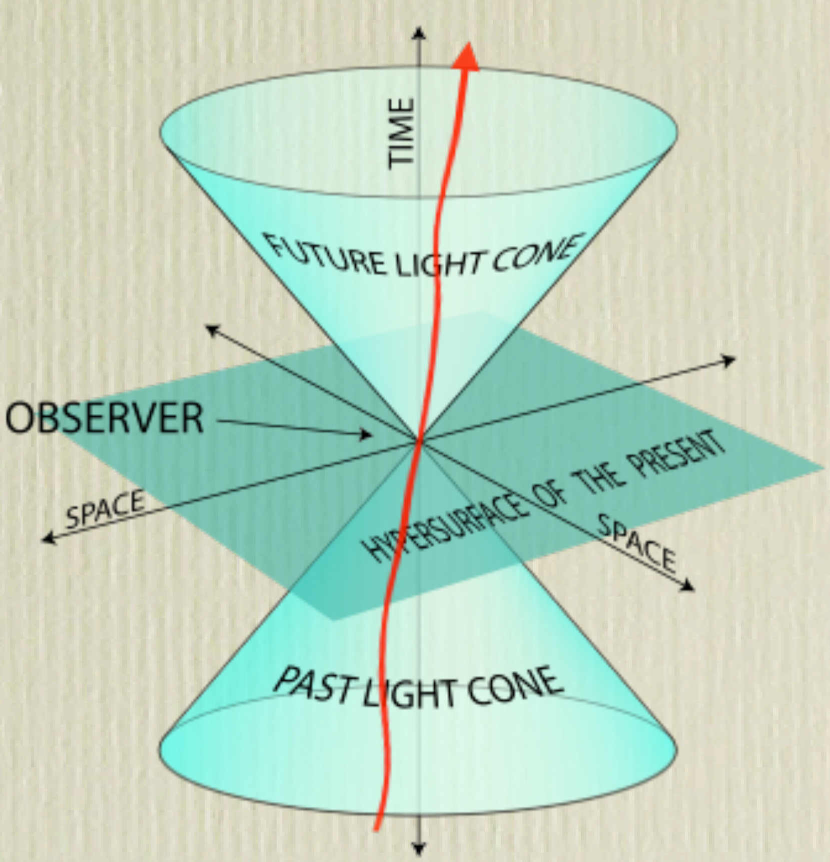
four-potential: $A^\alpha = (\phi, \vec{A}) \qquad \Longrightarrow \qquad F^{\alpha\beta} = \frac{\partial A^\beta}{\partial x_\alpha} - \frac{\partial A^\alpha}{\partial x_\beta}$

Covariant Formulation of Lorentz Force equation

In classical mechanics: $\vec{x} = (x(t), y(t), z(t))$

In special relativity: $x^\alpha = (x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau))$

The parameter τ is the proper time, the time measured in the reference frame of the particle itself.



The generalization of the velocity is the <u>four-velocity</u>:

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(1, \vec{v})$$

where $\gamma = \dfrac{1}{\sqrt{1-v^2}}$

The Lorentz force equation $m\dot{\vec{v}} = q(\vec{E} + \vec{v} \times \vec{B})$ turns into: $m\dot{u}^\alpha = qF^{\alpha\beta}u_\beta$

The Problem of Radiation-Reaction

The Lorentz force equation: $m\dot{u}^{\alpha} = -eF^{\alpha\beta}u_{\beta}$

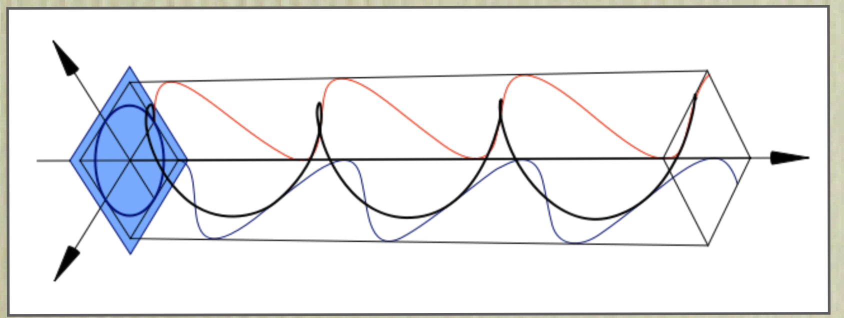
Maxwell equations => the rate at which energy is radiated away from the electron is:

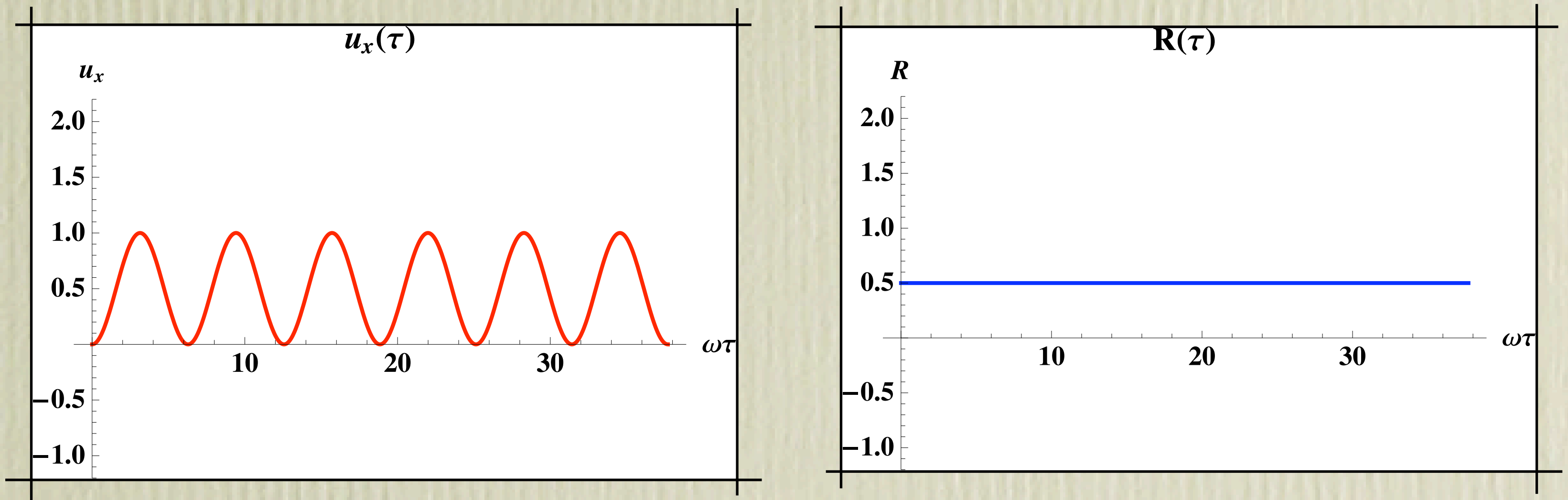
$$\mathcal{R} = -m\tau_0 \dot{u}^{\alpha}\dot{u}_{\alpha} \qquad \tau_0 = \frac{2}{3}\frac{e^2}{m} = 6.24 \times 10^{-24}\,\mathrm{s}$$

Therefore an accelerating charge loses energy.

This effect is not included in the Lorentz force equation!

<u>Example</u>: circularly polarized plane wave





Lorentz force equation doesn't account for energy lost

The Lorentz-Abraham-Dirac (LAD) Equation

Dirac (1938): Maxwell equations and energy conservation give

$$m\dot{u}^\alpha = \underbrace{-eF^{\alpha\beta}u_\beta}_{F_{Lorentz}} + \underbrace{m\tau_0 \left[\ddot{u}^\alpha + \dot{u}^2 u^\alpha\right]}_{F_{RR}}$$

This is the LAD equation (Lorentz-Abraham-Dirac).

The 3rd order time derivative requires another initial condition (initial acceleration), and results in infinitely many non-physical solutions...

Dirac replaced the additional condition with an "asymptotic condition". Instead of giving the initial acceleration, give the final acceleration.

The Landau-Lifshitz (LL) Equation

The LAD: $m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta + m\tau_0\left[\ddot{u}^\alpha + \dot{u}^2 u^\alpha\right]$

If τ_c is a characteristic time scale, the radiation-reaction force is of order $\varepsilon = \dfrac{\tau_0}{\tau_c}$

So to leading order in ε, we get the Lorentz force Eq:

$$m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta + O(\varepsilon)$$

Therefore,

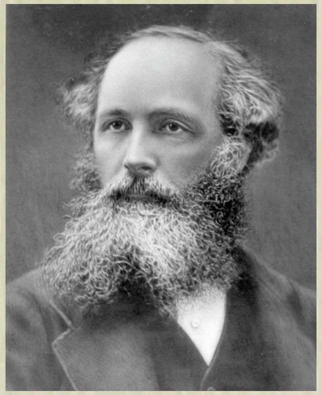
$$m\ddot{u}^\alpha = -e\frac{d}{d\tau}\left(F^{\alpha\beta}u_\beta\right) + O(\varepsilon)$$

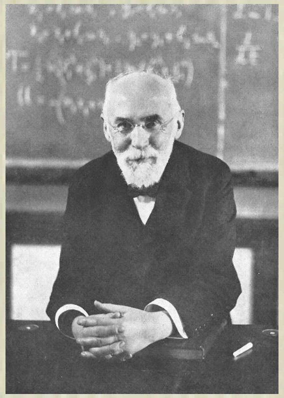
Using this approximation in the LAD gives the Landau-Lifshitz equation:

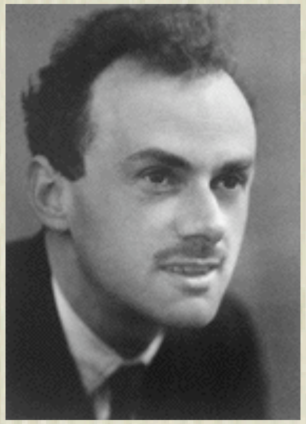
$$m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta - e\tau_0\left\{F^{\alpha\beta}_{,\gamma}u_\beta u^\gamma - e/m\left[F^{\alpha\beta}F_{\beta\gamma}u^\gamma - F^{\beta\gamma}F_{\gamma\delta}u^\delta u_\beta u^\alpha\right]\right\}$$

History of the Radiation-Reaction Problem

- 1865 : Maxwell formulates the theory of electromagnetism
- 1892 : Lorentz introduces the Lorentz force equation and argues that the radiation-reaction force is missing...
- 1903 : Abraham finds the radiation-reaction force
- 1904: Lorentz assumes that the electron is a solid sphere, and proves that the self-force exerted by the sphere on itself = radiation-reaction force
- 1905: Poincaré realizes that this means that in the lack of other forces, the sphere will explode...
- 1905-1920: People realize that the atom will be unstable
- 1920's: Schrödinger and his buddies formulate Quantum Mechanics. For about a decade people "forgot" about radiation-reaction...
- 1938: Dirac derives the covariant Lorentz-Abraham-Dirac equation
- 1938-1948: The physics community notices that the LAD has non-physical solutions
- 1948: Eliezer "derives" a new equation
- 1951: Landau & Lifshitz introduce the LL equation
- 1956: Caldirola claims that time is discrete, and replaces the ODE with a finite-difference equation making all the numerics people happy...
- 1962: Prigogine & Henin hypothesize a new equation







History of the Radiation-Reaction Problem

- 1964: Nodvik hypothesizes a new equation
- 1970: Teitelboim hypothesizes a new equation
- 1971: Mo & Papas modify energy-momentum conservation and hypothesize a new equation
- 1972: Leiter criticizes Mo & Papas. "One cannot modify the law of energy conservation..."
- 1973: Mo & Papas to Leiter "Only experiment will tell what is energy conservation"
- 1976: Gonzales & Gascon claim that LAD is only an approximated equation and derive a new equation.
- 1977: Petzold & Sorg generalize Caldirola's equation
- 1981: Valle et al. claim that Mo-Papa's equation is the correct equation
- 1988: Valentini proves the non-physical solutions of LAD are due to non-analytic fields
- 1992: Yaghjian derives an equation for a spherical particle
- 1997-2000: Rohrlich claims that Yaghjian's equation is the correct equation
- 2008: Rohrlich "Using physical arguments, I derive the physically correct equations of motion for a classical charged particle from the LAD equation which is well known to be physically incorrect."
- 2009: Gralla, Harte & Wald rederive Landau-Lifshitz equation <u>rigorously</u> using distribution theory
- 2009: Sokolov et al. introduce another equation

Mathematical Origins of the Problem of Radiation-Reaction

Maxwell's equations: $\partial_\alpha F^{\alpha\beta} = 4\pi J^\beta$

Lorentz force equation: $m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta$

There are three different kind of problems:

(1) Studying the evolution of the electromagnetic fields for known sources J^α: it has a well-posed initial value formulation.

Also for a point-particle: $J^\alpha(x) = -eu^\alpha\delta(x - z(t))\frac{d\tau}{dt}$

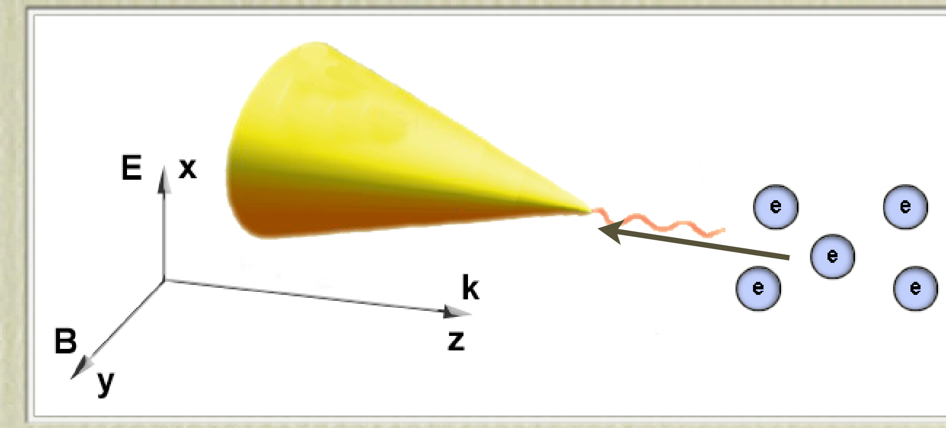
(2) Study the motion of a particle in a given external electromagnetic field $F^{\alpha\beta}$: it also has a well-posed initial value formulation.

(3) The coupled system: doesn't make mathematical sense because the field is necessarily singular at the position of the particle

The Radiation-Reaction Dominated Regime

Solve the Landau-Lifshitz equation analytically for a laser wave.

Recall that the rate at which energy is radiated is: $\mathcal{R} = -m\tau_0 \dot{u}^\alpha \dot{u}_\alpha$



for the solution of Landau-Lifshitz:

$$\mathcal{R} = -\frac{2}{3}e^2 \frac{(k\cdot u)^4}{(k\cdot u_0)^2}\left\{a_0^2 \hat{A}'^2 \quad \text{(Lorentz)} \right.$$
$$\left. +\tau_0 (k\cdot u_0)\left[2a_0^2 \hat{A}''\cdot\hat{A}' - 2a_0^4\psi\hat{A}'^2\right]\right\}$$

Landau-Lifshitz correction $\sim \omega\tau_0 a_0^4$

$a_0 = \dfrac{eA}{m}$ intensity of laser

$\hat{A}^\alpha$ normalized 4-potential

k^α wave 4-vector

u_0^α initial 4-velocity

For a typical laser $\omega\tau_0 \sim 10^{-8}$

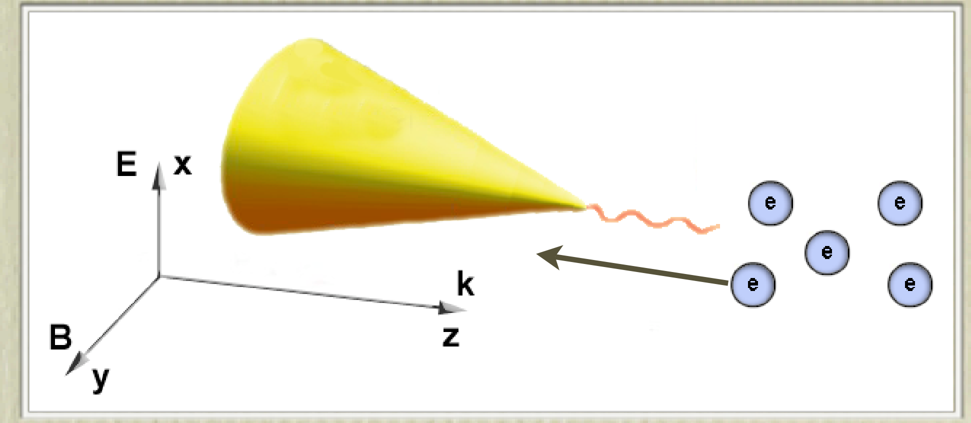
So Radiation-reaction effects are important when: $a_0^2 \sim 10^8$

This is far beyond current technological capabilities $a_0 \sim 10$

The Radiation-Reaction Dominated Regime

$$u^\alpha = \gamma(1, \vec{v}) \Longrightarrow \ u^\alpha = O(\gamma)$$

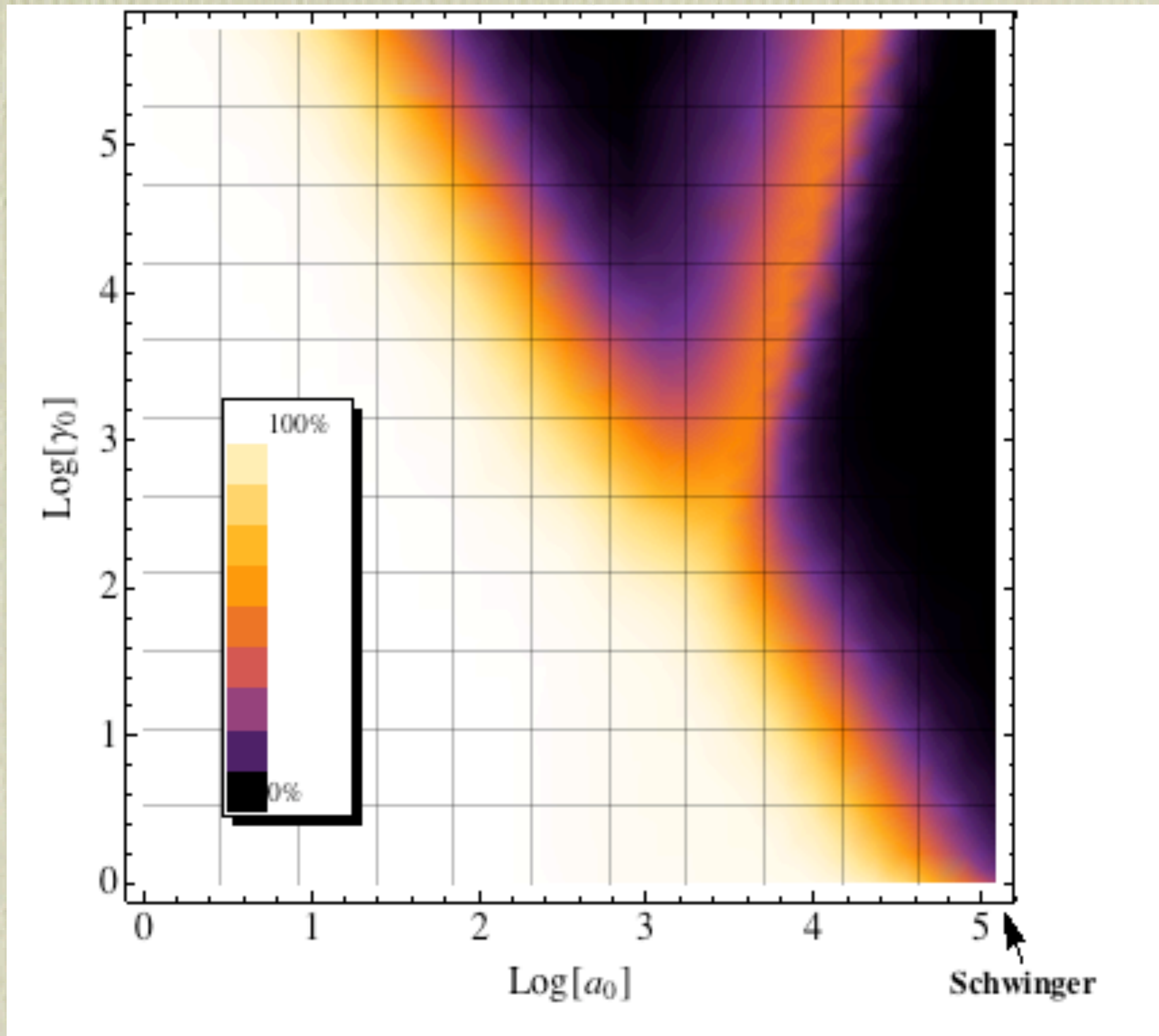
This scales radiation-reaction effects by a factor γ_0!



RRDR Criterion:

$$\gamma_0 a_0^2 \sim 10^8$$

$$\Delta = \frac{1}{2\pi}\int_0^{2\pi} \frac{|E_{LL}(\xi) - E_{Lorentz}(\xi)|}{|E_{LL}(\xi) + E_{Lorentz}(\xi)|}$$



Collaboration

This work was done under the supervision of Prof. Johann Rafelski from the physics department, together with:

- Lance Labun (Physics department UoA)
- N. Elkina, C. Klier & H. Ruhl (LMU Munich)

Thank you!

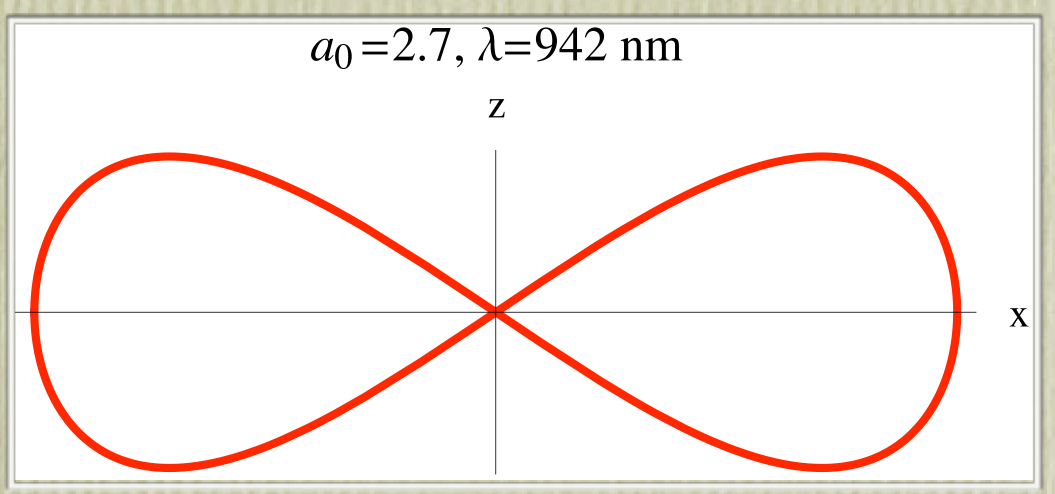
Pictures for Linearly Polarized Wave

$a_0 = 2.7 \quad T = 26.8\,\mathrm{fs}$

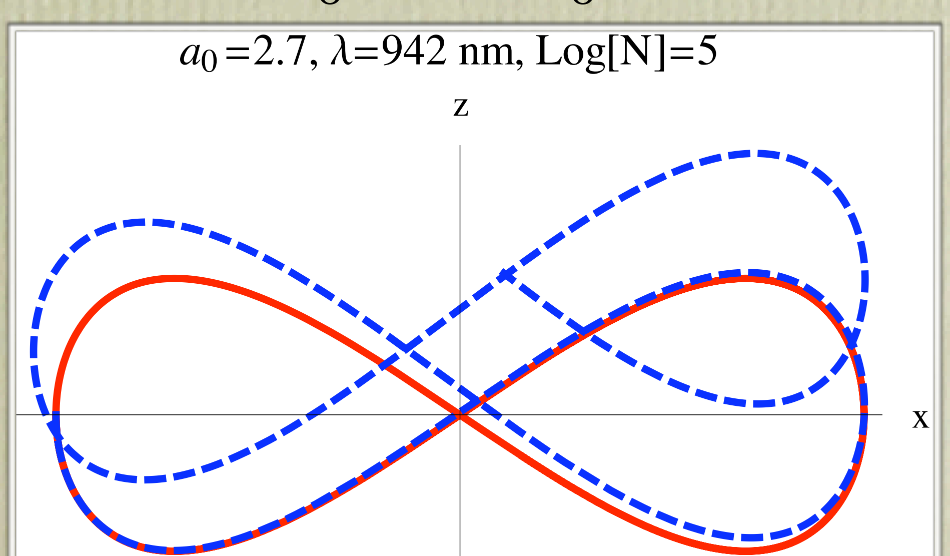
$\lambda = 942\,\mathrm{nm} \quad \omega = 2\,\mathrm{fs}^{-1}$

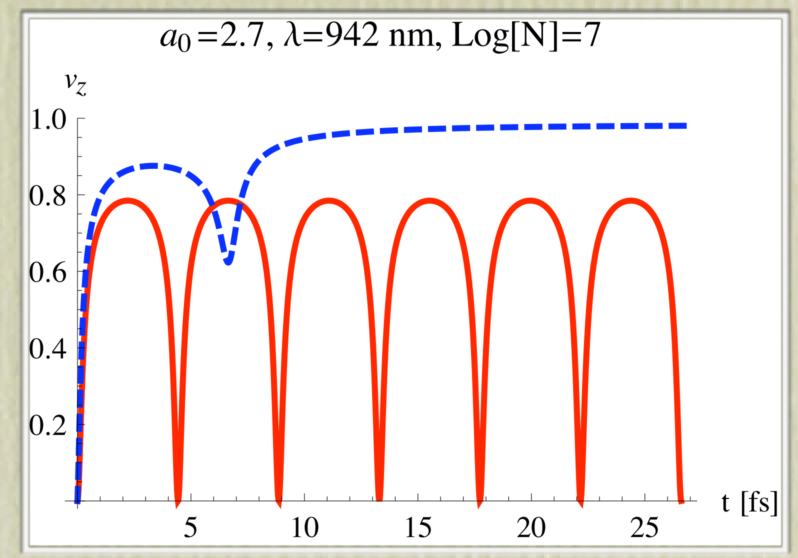
Wave Direction: $\hat{z}$ Polarization: $\hat{x}$

Lorentz Landau-Lifshitz

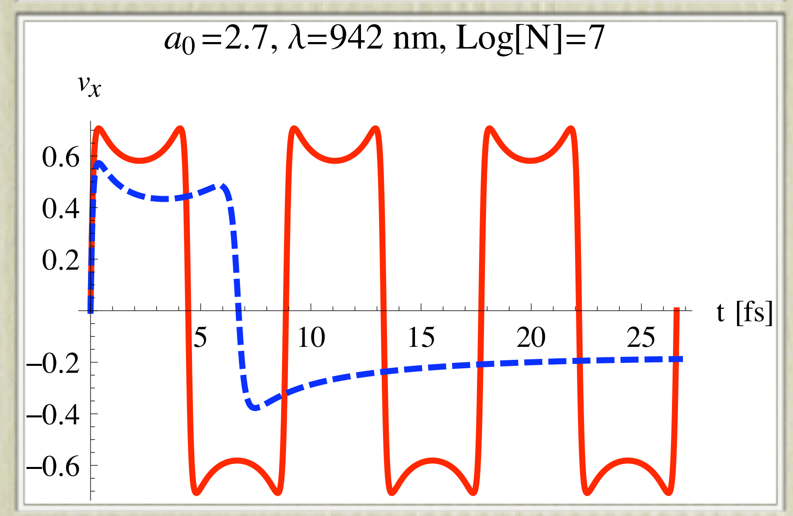


$\tau_0 \mapsto N\tau_0$

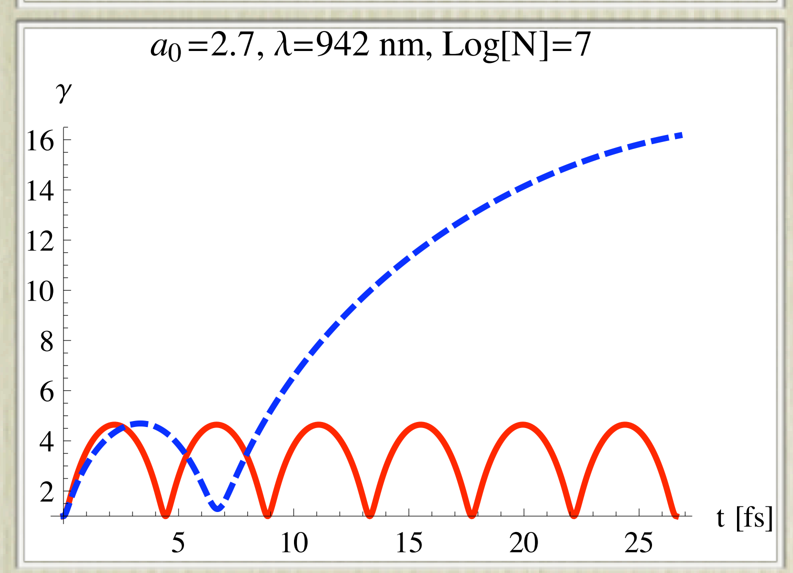




Momentum gain in direction of wave

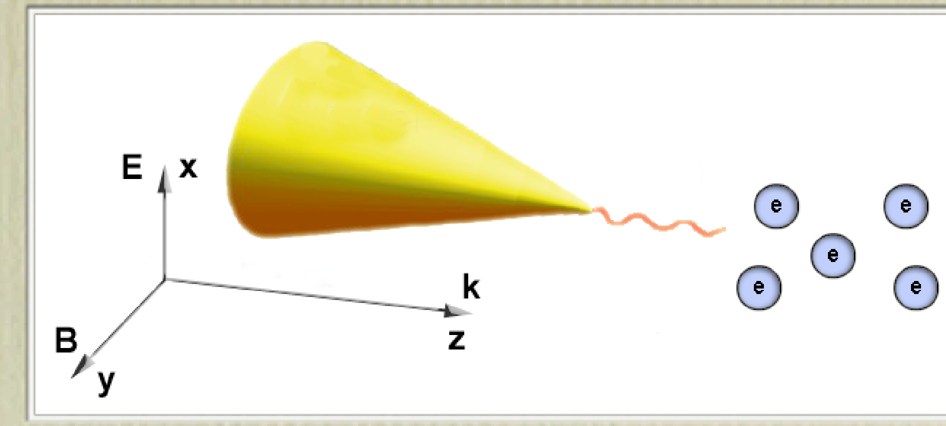


Momentum loss in direction of polarization



But altogether each electron in the cluster gains more energy

The Setup



We solve the LL eq. for the potential:

$$A^\alpha(x) = A_0 \mathrm{Re}\left[\varepsilon^\alpha f(\xi)\right]$$

with:

- Polarization vector ε^α
- Wave vector k^α
- $\xi = k \cdot x$

$$\left.\begin{aligned} k^2 &= 0 \\ |\varepsilon|^2 &= -1 \\ k \cdot \varepsilon &= 0 \end{aligned}\right.$$

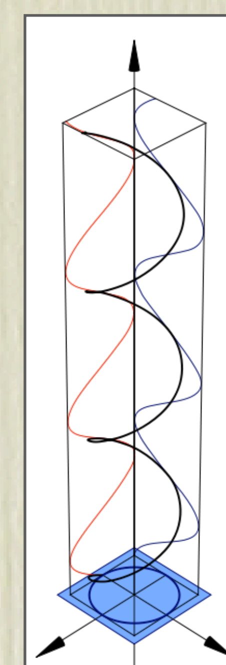
Examples

Linear Polarization

$$\begin{aligned} \varepsilon^\alpha &= (0,1,0,0) \\ k^\alpha &= (\omega, 0, 0, k) \\ f(\xi) &= \sin(\xi) \\ \vec{A} &= -A_0 \sin(kz - \omega t)\hat{x} \end{aligned}$$

Circular Polarization

$$\begin{aligned} \varepsilon^\alpha &= \frac{1}{\sqrt{2}}(0,1,-i,0) \\ k^\alpha &= (\omega, 0, 0, k) \\ f(\xi) &= \sqrt{2}e^{i(\xi - \xi_0)} \\ \vec{A} &= A_0\left[\cos(kz - \omega t)\hat{x} \right. \\ &\quad \left. - \sin(kz - \omega t)\hat{y}\right] \end{aligned}$$



Solution of the Landau-Lifshitz Equation

The LL equation is non-linear and therefore very difficult to solve in terms of proper time...

$$m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta - e\tau_0\left\{F^{\alpha\beta}_{,\gamma}u_\beta u^\gamma - e/m\left[F^{\alpha\beta}F_{\beta\gamma}u^\gamma - F^{\beta\gamma}F_{\gamma\delta}u^\delta u_\beta u^\alpha\right]\right\}$$

"Trick": change variables $\tau \mapsto \xi = k \cdot x$ $\qquad \dfrac{d\xi}{d\tau} = k \cdot u$

$$(k\cdot u)\, u'^\alpha \quad = \quad -\frac{e}{m}F^{\alpha\beta}u_\beta - \frac{e}{m}\tau_0\left\{F^{\alpha\beta}_{,\gamma}u_\beta u^\gamma - \frac{e}{m}\left[F^{\alpha\beta}F_{\beta\gamma}u^\gamma - u_\beta F^{\beta\gamma}F_{\gamma\delta}u^\delta u^\alpha\right]\right\}$$

SOLUTION