PROBING RADIATION REACTION IN THE HIGH ACCELERATION REGIME

by

Yaron Hadad

Based on a work with
L. Labun, J. Rafelski, N. Elkina, C. Klier and H. Ruhl
OUTLINE

• The problems of radiation reaction and self force
• Suggested models
• Solution of the Landau-Lifshitz equation
• Probing the radiation reaction problem
THE PROBLEM OF RADIATION REACTION

The Lorentz Force (LF) Eq. \[ m \dot{u}^\alpha = -e F_{\text{ext}}^{\alpha \beta} u_\beta \]

The rate at which energy is radiated away from the electron is:

\[ \mathcal{R} = -m \tau_0 \dot{u}^\alpha \dot{u}_\alpha \]

\[ \tau_0 = \frac{2 e^2}{3 m} = 6.24 \times 10^{-24} \text{ s} \]

\[ \implies \text{An accelerating charge loses energy.} \]

This effect is not included in the Lorentz force equation.

The rate at which the energy-momentum of radiation is emitted is

\[ \frac{dP^\alpha}{d\tau} = \mathcal{R} u^\alpha \]
THE PROBLEM OF SELF FORCE

Problem #1: Motion of a particle in a known external field

\[ m \ddot{u}^\alpha = -e F^\alpha_\beta \text{ext} u_\beta \] well-posed for any external field \( F^\alpha_\beta \text{ext} \)

Problem #2: Evolution of the field for known currents

\[ \partial_\alpha F^\alpha_\beta = 4\pi J^\beta \] well-posed if \( \partial_\alpha J^\alpha = 0 \)

Also well-defined for a point particle

\[ J^\alpha (x) = -e u^\alpha \delta(x - z(t)) \frac{d\tau}{dt} \]

Problem #3 = #1 + #2: The coupled system

Mathematically ill-defined, experimentally untested
Classical theory of radiating electrons

By P. A. M. Dirac, F.R.S., St John’s College, Cambridge

(Received 15 March 1938)

It may be wondered why this problem was not solved long ago. A great deal of work has been done in the past in
THE LORENTZ-ABRAHAM-DIRAC EQUATION

Maxwell + energy conservation + simplicity $\Rightarrow$
The LAD Eq.

\[
m\dot{u}^\alpha = -eF^{\alpha\beta} u_\beta + m\tau_0 \left[ \ddot{u}^\alpha + \dot{u}^2 u^\alpha \right]
\]

Requires another initial condition (acceleration)

$\Rightarrow$ Get non-physical solutions

To get physical solutions, Dirac replaced the initial acceleration with final acceleration.

The LAD has runaway solutions and no action principle.
THE LORENTZ-ABRAHAM-DIRAC EQUATION

Maxwell + energy conservation + simplicity $\implies$

The LAD Eq.

$$m \ddot{u}^\alpha = -e F_{\alpha\beta} u_{\beta} + m \tau_0 \left[ \dddot{u}^\alpha + \dot{u}^2 u^\alpha \right]$$

- external force
- self force
- radiation-reaction

Schott

Equations which are already well known. It may be wondered why this problem was not solved long ago. A great deal of work has been done in the past in

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Dirac 1938
For all practical purposes $F_{\text{self force}} \ll F_{\text{Lorentz}}$

In the leading order

$$m \dot{u}^\alpha = -e F^{\alpha \beta} u_\beta + m \tau_0 \left[ \ddot{u}^\alpha + \dot{u}^2 u^\alpha \right]$$

Differentiate to eliminate third derivative

$\implies$ get the Landau-Lifshitz (LL) equation:

$$m \dot{u}^\alpha = -e F^{\alpha \beta} u_\beta - e \tau_0 \left[ F_{,\gamma}^{\alpha \beta} u_\beta u^\gamma - \frac{e}{m} \left( F^{\alpha \beta} F_{\beta \gamma} u^\gamma - F^{\beta \gamma} F_{\gamma \delta} u^\delta u_\beta u^\alpha \right) \right]$$

The LL equation is nonlinear in the field and four-velocity
## Suggested Models

<table>
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**HERBERT SPOHN**

_Zentrum Mathematik and Physik Department, TU München, D-80290 München, Germany_

actually a surface of the form $\dot{x} = h(x, \dot{x})$. Thus, for given initial conditions $x(0), \dot{x}(0)$, there is exactly one solution on the critical surface and, as to be shown, it satisfies the asymptotic condition. (2) There is an effective second order equation, given below, which governs the motion on the critical surface. Thus the initial value problem is restored and the equation can

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**Landau-Lifshitz (1952)**

\[ m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta - e\tau_0 \left[ F^{\alpha\gamma\beta}u^\gamma - \frac{e}{c} \left( F^{\alpha\beta\gamma}F_{\beta\gamma} - F^{\beta\gamma}F_{\gamma\delta}u^\delta u^\beta u^\gamma \right) \right] \]

**A Rigorous Derivation of Electromagnetic Self-force**

Samuel E. Gralla, Abraham I. Harte, and Robert M. Wald

*Enrico Fermi Institute and Department of Physics*

*University of Chicago*

We considered a one-parameter-family of solutions to the Maxwell and matter equations (1)-(3) and (7) containing a body that “shrinks down” to zero size, mass, and charge according to the scaling assumptions of section [II]. We found that the lowest-order perturbative equations. In the case of negligible spin and electromagnetic dipole moment, this reduces to the reduced-order ALD equation.

Consider a transverse wave:

\[ A^\alpha(x) = A_0 \Re \left[ \varepsilon^\alpha f(\xi) \right] \]

\[ \varepsilon^\alpha = (0, 0, 1, 0) \]
\[ k^\alpha = (\omega, k, 0, 0) \]
\[ f(\xi) = A_0 \cos(\xi) \]
\[ \vec{A} = A_0 \cos(\xi)\hat{y} \]

**Linear polarization**

**Circular polarization**

\[ \varepsilon^\alpha = \frac{1}{\sqrt{2}} (0, 0, 1, \pm i) \]
\[ k^\alpha = (\omega, k, 0, 0) \]
\[ f(\xi) = A_0 e^{i\xi} \]
\[ \vec{A} = \frac{A_0}{\sqrt{2}} [\cos(\xi)\hat{y} \mp \sin(\xi)\hat{z}] \]
SOLVING THE LANDAU-LIFSHITZ EQUATION

\[ m\dot{u}^\alpha = -eF^{\alpha\beta}u_\beta - e\tau_0 \left[ F^{\alpha\beta}_{\gamma}\dot{u}_\beta u^\gamma - \frac{e}{m} (F^{\alpha\beta}F_{\beta\gamma}\dot{u}^\gamma - F^{\beta\gamma}F_{\gamma\delta}\dot{u}^\delta u_\beta u^\alpha) \right] \]

Equation is **nonlinear** and contains \( k \cdot u \) and \( \varepsilon \cdot u \)

\[ \implies \text{coupling between the 4-velocity's components} \]

Change variables \( \tau \rightarrow \xi = k \cdot x \) and contract with \( k^\alpha \)

\[ k \cdot u = \frac{k \cdot u_0}{1 - \tau_0a_0^2(k \cdot u_0)\psi(\xi)} \]  \( \begin{cases} \text{LF} & k \cdot a = 0 \\ \text{LL} & k \cdot a \neq 0 \end{cases} \)

\[ a_0 = \frac{eA_0}{m} \quad \psi(\xi) = \int_0^\xi \left[ \hat{A}'(y) \right]^2 dy \]

Similarly, contract with \( \varepsilon^\alpha \) \( \implies \) get \( \varepsilon \cdot u \) \( \implies \) get \( u^\alpha \)

solution is quite complicated
THE RADIATION-REACTION DOMINATED REGIME

The rate at which energy is radiated away from the electron is:

\[
\mathcal{R} = -\frac{2}{3} e^2 \frac{(k \cdot u)^4}{(k \cdot u_0)^2} \left\{ a_0^2 \hat{A}'^2 \right. \\
+ 2(k \cdot u_0) \tau_0 \left[ a_0^2 \hat{A}' \cdot \hat{A}'' - a_0^4 \Psi \hat{A}'^2 \right] + O(\tau_0^2) \left. \right\} \\
\]

\[
k \cdot u_0 = \gamma_0 (\omega - \vec{k} \cdot \vec{v}_0)
\]

Radiation-reaction is important if \( a_0^2 \sim (\omega \tau_0) \gamma_0 a_0^4 \)

\[
\lambda \sim 942 \text{ nm} \quad \omega \sim 2 \text{ fs}^{-1}
\]

RRDRR Criterion: \( \gamma_0 a_0^2 \sim 10^8 \)
PROBING THE RADIATION-REACTION PROBLEM

\[ \Delta = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{|E_{LL}(\xi) - E_{\text{Lorentz}}(\xi)|}{|E_{LL}(\xi) + E_{\text{Lorentz}}(\xi)|} d\xi \]

Different LL and LAD dynamics

Critical acceleration

\[ a_c = \frac{mc^3}{\hbar} \]

\[ a_c = 2.3 \times 10^{29} \frac{m}{s^2} \]
PROBING THE RADIATION-REACTION PROBLEM

Linearly polarized wave

\( \alpha_0 = 100 \)

\( \gamma_0 = 1 \)

\( \gamma_0 = 1,000 \)
WHAT ABOUT ACCELERATION?

Circularly polarized wave

\[ \gamma_0 = 1,000 \quad a_0 = 100 \quad E_e = 0.511 \text{ GeV} \]

\[ a = \sqrt{-\dot{u}^\alpha \dot{u}_\alpha} \]

Can acceleration be arbitrarily high?
Thank you!