Gravitational Solitons on Einstein-Rosen Metrics

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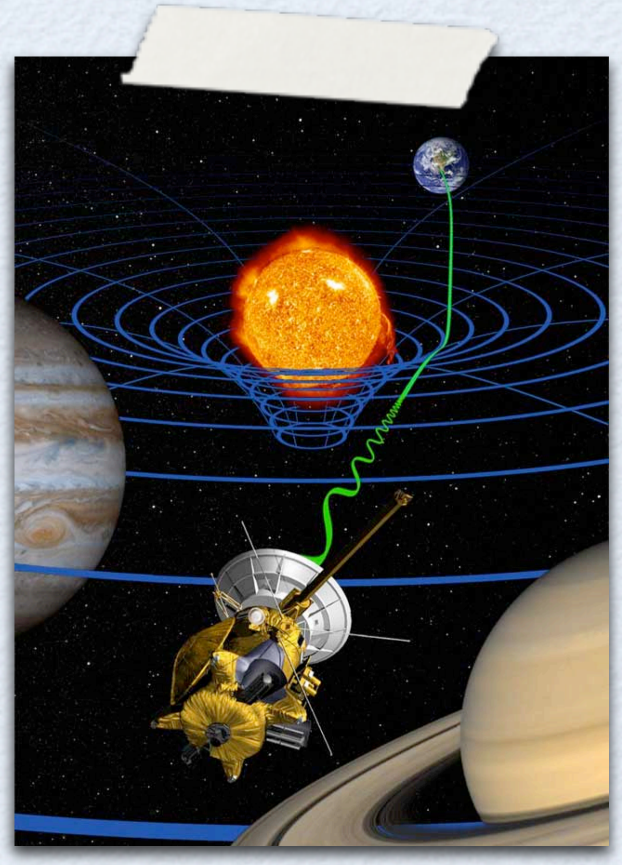
Based on the work
'Gravitational Solitons on Diagonal Background Metrics and on Einstein-Rosen Background' with V. E. Zakharov (in process)

The Theory of Relativity

<u>The Equivalence Principle</u>: gravity affects all bodies in the same way, independently of the body's composition.



The dynamic field is the spacetime metric: $g_{\mu\nu}$

Spacetime curves in the presence of matter according to Einstein's equation:

$$\underbrace{R_{\mu\nu}}_{\textbf{Ricci Curvature}} - \frac{1}{2} g_{\mu\nu} \underbrace{R}_{\textbf{Scalar Curvature}} = (8\pi G) \underbrace{T_{\mu\nu}}_{\textbf{Stress-Energy Tensor}} \qquad (c = 1)$$

In the absence of matter we get Einstein's vacuum equation:

$$R_{\mu\nu} = 0$$

The Setup

The spacetime metric in matrix form:

$$g_{\mu\nu} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

We get the spacetime interval:

$$-ds^2 = f(-dt^2 + dz^2) + g_{ab}dx^a dx^b \quad (a, b = 1, 2)$$

Physical assumptions:

$$g_{13} = g_{23} = 0$$

$$f = f(t, z)$$

$$g_{ab} = g_{ab}(t, z)$$

Includes Schwarzschild metric, Kerr metric, Einstein-Rosen metric, Bianchi metrics, Kasner metric and many more...

Einstein's Equation

Use 'light-cone' coordinates: $t = \zeta - \eta \qquad z = \zeta + \eta$

Einstein's vacuum equation:

$$R_{\mu\nu} = 0$$

$$R_{ab} = 0$$

$$\underbrace{\left(\alpha g_{,\zeta} g^{-1}\right)}_{-A}{}_{,\eta} + \underbrace{\left(\alpha g_{,\eta} g^{-1}\right)}_{B}{}_{,\zeta} = 0$$

$$\det\left(g\right) = \alpha^2$$

$$R_{00} + R_{33} = 0$$
$$R_{03} = 0$$

$$\left(\ln f\right)_{,\zeta} = \frac{\left(\ln \alpha\right)_{,\zeta\zeta}}{\left(\ln \alpha\right)_{,\zeta}} + \frac{1}{4\alpha\alpha_{,\zeta}} Tr\left(A^2\right)$$

$$\left(\ln f\right)_{,\eta} = \frac{\left(\ln \alpha\right)_{,\eta\eta}}{\left(\ln \alpha\right)_{,\eta}} + \frac{1}{4\alpha\alpha_{,\eta}} Tr\left(B^2\right)$$

$$\alpha_{,\zeta\eta} = 0 \implies \alpha\left(\zeta, \eta\right) = a\left(\zeta\right) + b\left(\eta\right) \qquad \beta\left(\zeta, \eta\right) = a\left(\zeta\right) - b\left(\eta\right)$$

Belinski & Zakharov (1978) found a Lax pair for the $g(\zeta, \eta)$ eq.

The Inverse Transform

1

Take a particular solution $g_0\left(\zeta,\eta\right)$

2

Can skip this step when g_0 is diagonal

$$
\begin{aligned}
D_1\psi_0 &= \frac{A_0}{\lambda-\alpha}\psi_0 \\
D_2\psi_0 &= \frac{B_0}{\lambda+\alpha}\psi_0
\end{aligned}
\quad \Longrightarrow \text{ Get } \psi_0
$$

3

Define the dressing matrix:

$$\chi = I + \sum_{k=1}^{n}\left(\frac{R_k}{\lambda-\mu_k}+\frac{\bar{R}_k}{\lambda-\bar{\mu}_k}\right)$$

$$\mu_k = \omega_k-\beta\pm\sqrt{\left(\omega_k-\beta\right)^2-\alpha^2}$$

R_k can be expressed in terms of ψ_0

4

Get a solitonic solution!

$$g=\chi\left(0\right)g_0$$

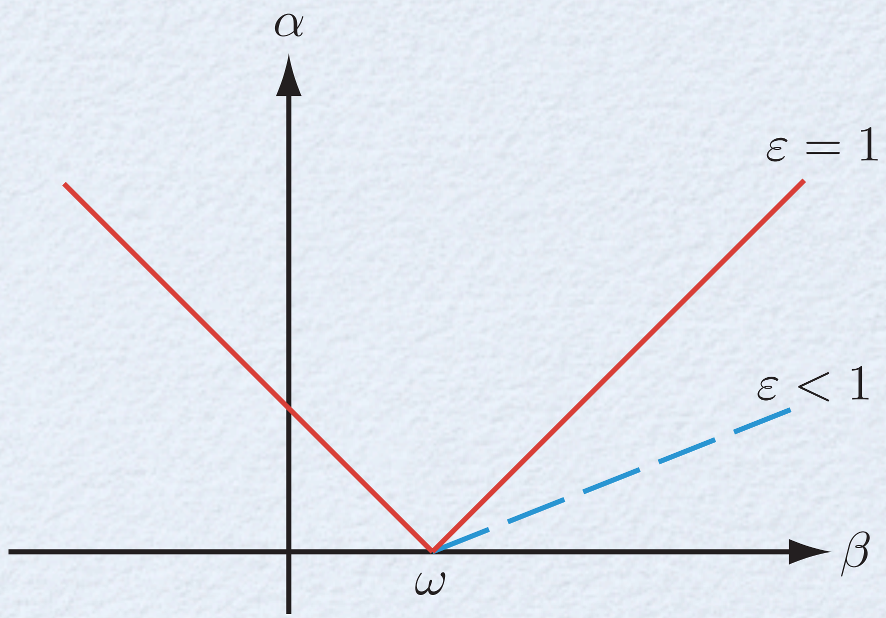
Diagonal Metrics

$$g_0 = \mathrm{diag}(\alpha e^{2s}, \alpha e^{-2s})$$

$\Longrightarrow$ n-soliton solution can be written explicitly in this case, but will depend on the functions ρ_k:

Cannot always be integrated analytically.

$$\begin{cases} \rho_{k,\zeta} = \dfrac{\alpha + \mu_k}{\alpha - \mu_k} 2s_{,\zeta} \\ \rho_{k,\eta} = \dfrac{\alpha - \mu_k}{\alpha + \mu_k} 2s_{,\eta} \end{cases}$$



$$\mu_k = \omega_k - \beta \pm \sqrt{(\omega_k - \beta)^2 - \alpha^2}$$
$$= (\beta - \omega_k)(-1 \pm \sqrt{1 - \varepsilon_k^2})$$

$$\varepsilon_k = \frac{\alpha}{\beta - \omega_k}$$

Write: $\rho_k = \rho_k^{(0)} + \rho_k^{(1)} \varepsilon_k + \rho_k^{(2)} \varepsilon_k^2 + \dots$

$\Longrightarrow$ Get a recursive relation for ρ_k in terms of s only!

One-Soliton on Einstein-Rosen Background

Einstein-Rosen metric:

$$\varepsilon_k = \frac{r}{t-\omega_k}$$

$$t \mapsto r$$
$$z \mapsto t$$
$$x \mapsto i\theta$$
$$y \mapsto iz$$

$$g_0 = \mathrm{diag}(\alpha e^{2s}, \alpha e^{-2s})$$
$$\alpha(r,t) = r$$
$$s(r,t) = J_0(r)\sin(t)$$

$$\begin{aligned} \rho_k^{(0)}(r,t) &= 2J_0(r)\sin(t) \\ \rho_k^{(1)}(r,t) &= -2J_1(r)\cos(t) \\ &\vdots \end{aligned}$$

One-soliton solution:

$$g = \frac{1}{2\mu_1\cosh\rho_1}\begin{pmatrix} (\mu_1^2 e^{\rho_1} + \alpha^2 e^{-\rho_1})e^{2s} & \alpha^2-\mu_1^2 \\ \alpha^2-\mu_1^2 & (\alpha^2 e^{\rho_1} + \mu_1^2 e^{-\rho_1})e^{-2s} \end{pmatrix}$$

$$f = f_0\sqrt{\alpha}\mu_1\cosh\left(\frac{\rho_1}{\alpha^2-\mu_1^2}\right)$$

In this case the functions ρ_k are given by:

$$\rho_k(r,t) = 2J_0(r)\sin(t) - 2J_1(r)\cos(t)\varepsilon_k + \left[J_0(r)\sin(t) - \frac{2J_1(r)\sin(t)}{r}\right]\varepsilon_k^2 + O(\varepsilon_k^3)$$

Pretty Soliton

$$g = \frac{1}{2\mu_1 \cosh \rho_1} \begin{pmatrix} (\mu_1^2 e^{\rho_1} + \alpha^2 e^{-\rho_1}) e^{2s} & \alpha^2 - \mu_1^2 \\ \alpha^2 - \mu_1^2 & (\alpha^2 e^{\rho_1} + \mu_1^2 e^{-\rho_1}) e^{-2s} \end{pmatrix}$$

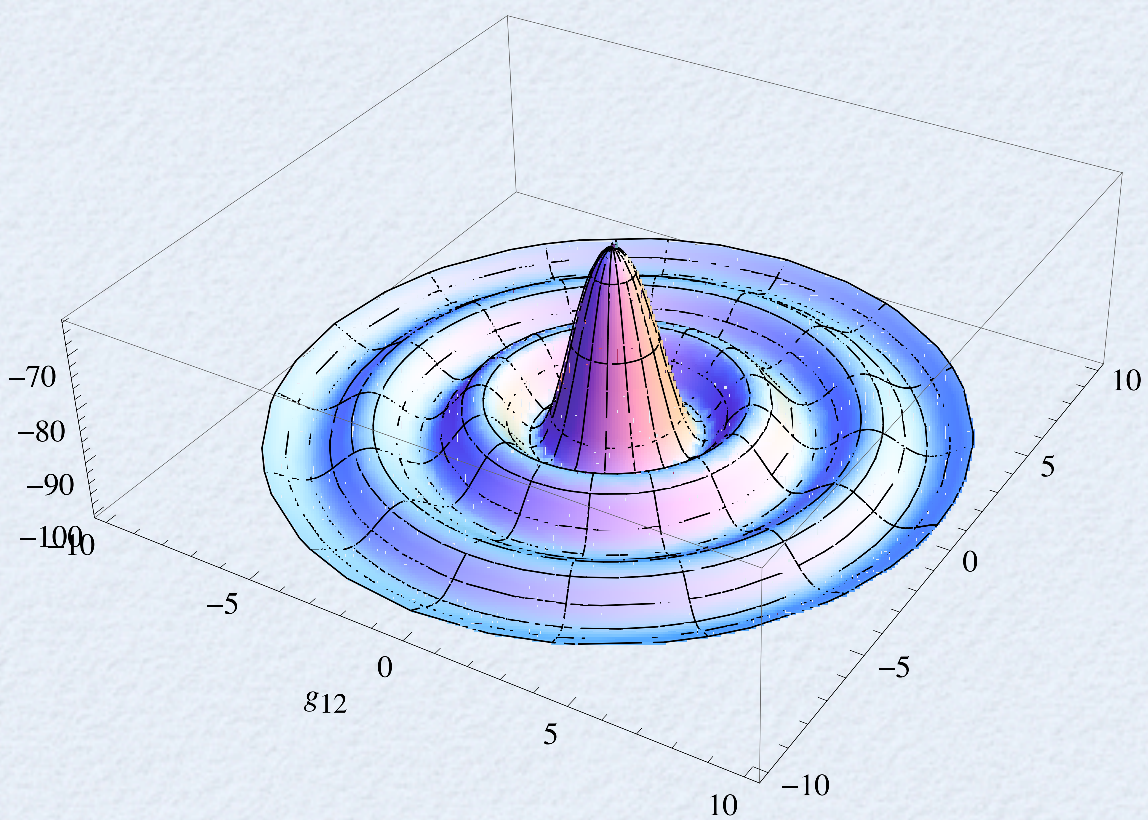
$$\omega_1 = 0$$

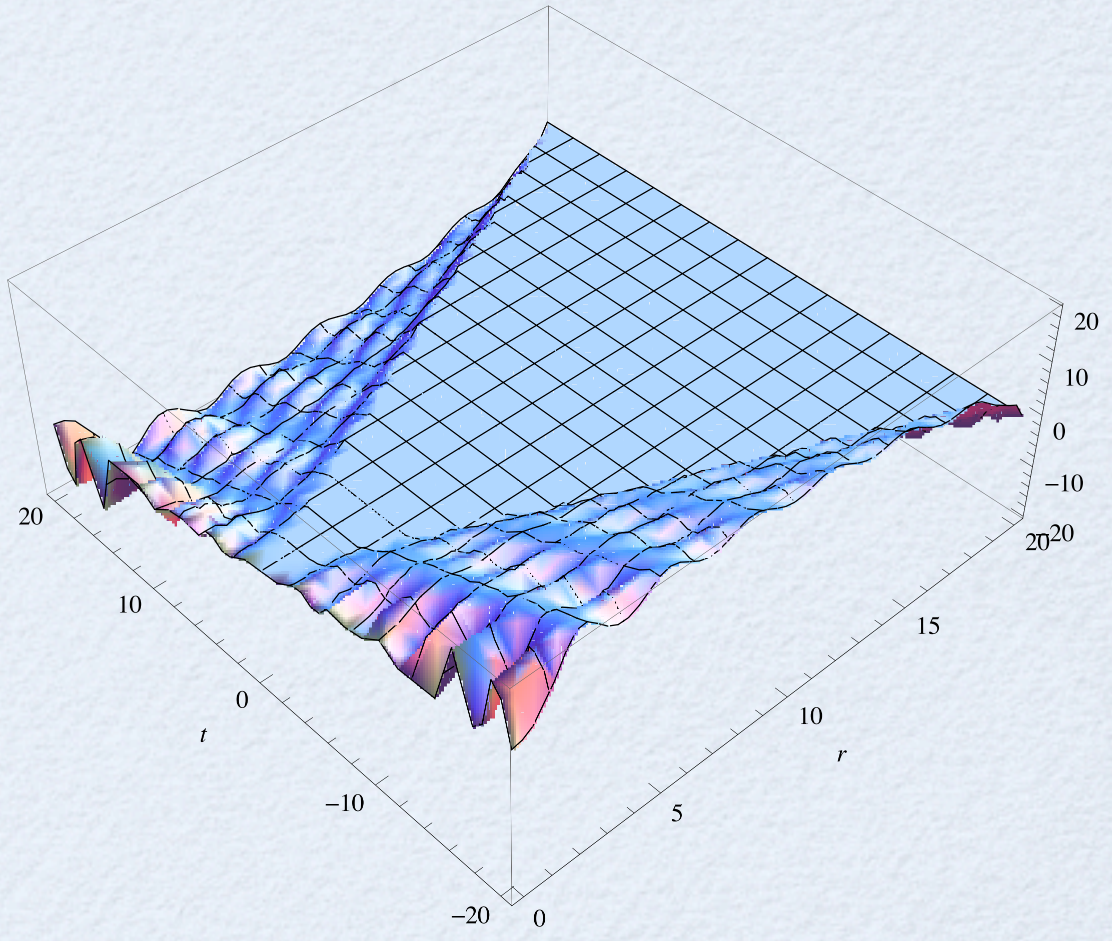
$$0 \leq r \leq 10 \qquad t = 100$$

What happens in the limit $\varepsilon \to 1$?

$$g \to g_0$$

$\Longrightarrow$ Solution can be extended.





Two Solitons

In this case there are two integration constants: $\omega_1, \omega_2 \in \mathbb{R}$

The light-cones divided spacetime into 6 regions:

- I, VI : The two-soliton region
- III : The two-soliton 'interaction' region
- II, V : One-Soliton regions
- IV : Background metric

$$\varepsilon_k = \frac{r}{t - \omega_k}$$