Gravitational Solitons

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Based on the work

'Integration of the Einstein equations by Means of the Inverse Scattering Problem
Technique and Construction of Exact Soliton Solutions'
by V. A. Belinski and V. E. Zakharov

Outline

- The Gist of the Inverse Scattering Transform
- The Theory of Relativity
- The Integration Scheme
- Gravitational Solitons
- Example: Solitons on Einstein-Rosen Background
- Future Goals

The Gist of the Inverse Scattering Transform

The Fourier Transform

$$\begin{tikzcd}
\text{linear PDE} \arrow[r, "\mathcal{F}"] \arrow[d, dashed] & \text{ODE} \arrow[d] \\
\text{solution} & \mathcal{F}[\text{solution}] \arrow[l, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

$$f \in L^1$$

$$\mathcal{F}[f](k) = \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

The Inverse Scattering Transform

$$\begin{tikzcd}
\text{nonlinear PD} \arrow[r] \arrow[d, dashed] & \text{2 linear PDEs} \arrow[d] \\
\text{solution} & \text{solution} \arrow[l]
\end{tikzcd}$$

Get Solitons!

Example #1

Example #2

Example #3

Special Relativity

Postulate: The speed of light c is the same for all observers.

$$c(dt) = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

Introduce the notation: $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$

and the matrix:

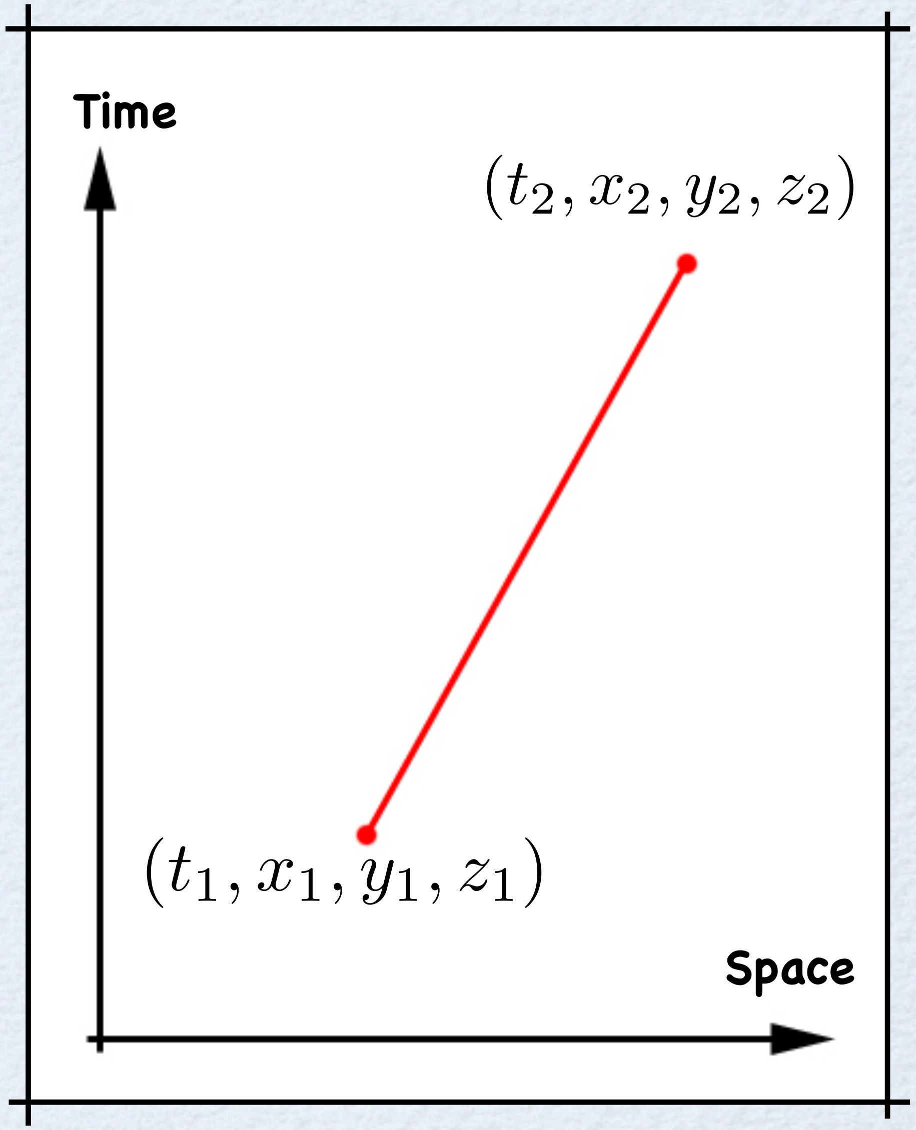
$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we can write the condition as:

$$-ds^2 := \ \eta_{\alpha\beta} dx^\alpha dx^\beta = 0$$

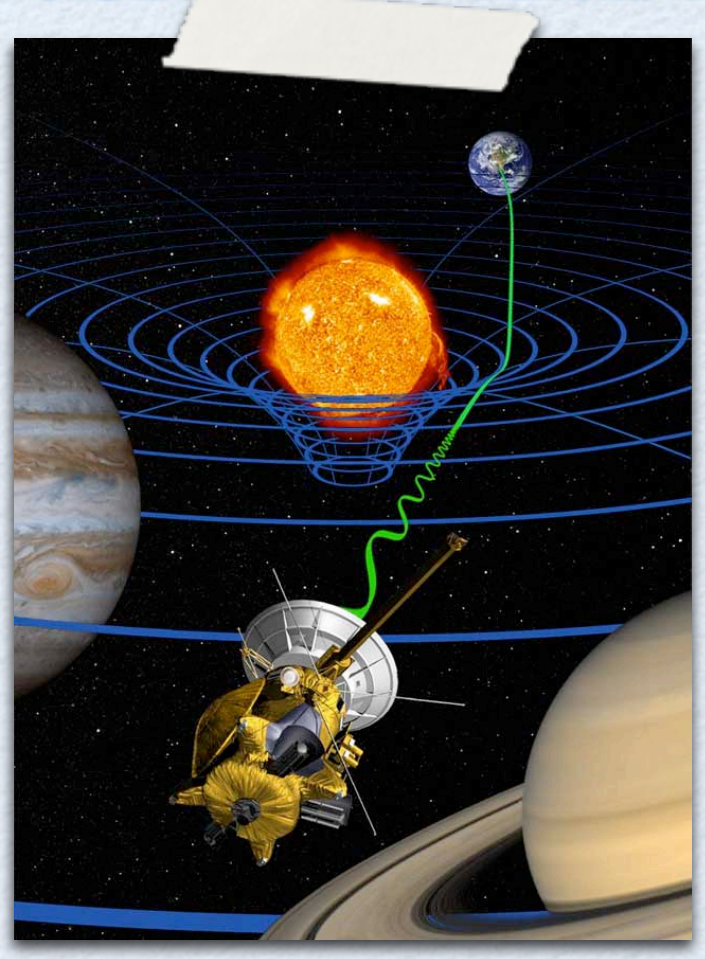
(sum over repeated indices)

Mathematically, this turns spacetime into a 4D manifold



General Relativity

The Equivalence Principle: gravity affects all bodies in the same way, independently of the body's composition.



The spacetime interval: $-ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

Spacetime curves in the presence of matter according to Einstein's equation:

$$\underbrace{R_{\alpha\beta}}_{\textbf{Ricci Curvature}} - \frac{1}{2} g_{\alpha\beta} \underbrace{R}_{\textbf{Scalar Curvature}} = \cancel{(8\pi G)} \underbrace{\cancel{T_{\alpha\beta}}}_{\textbf{Stress-Energy Tensor}} \qquad (c = 1)$$

In the absence of matter we get Einstein's vacuum equation:

$$R_{\alpha\beta} = 0$$

The Setup

The spacetime metric in matrix form:

$$g_{\alpha\beta} = \begin{pmatrix} \enclose{circle}{g_{00}} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & \enclose{circle}{g_{33}} \end{pmatrix}$$

The Setup

The spacetime metric in matrix form:

$$g_{\alpha\beta} = \begin{pmatrix} -f & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & f \end{pmatrix}$$

The Setup

The spacetime metric in matrix form:

$$g_{\alpha\beta} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & g_{13} \\ 0 & g_{21} & g_{22} & g_{23} \\ 0 & g_{31} & g_{32} & f \end{pmatrix}$$

The Setup

The spacetime metric in matrix form:

$$g_{\alpha\beta} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & g_{11} & g_{12} & 0 \\ 0 & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & f \end{pmatrix}$$

We get the spacetime interval:

$$-ds^2 = f(-dt^2 + dz^2) + g_{ab}dx^a dx^b \quad (a, b = 1, 2)$$

Actual assumptions:

$$g_{13} = g_{23} = 0$$

$$f = f(t, z)$$

$$g_{ab} = g_{ab}(t, z)$$

Includes Schwarzschild metric, Kerr metric, Einstein-Rosen metric, Bianchi metrics, Kasner metric and many more...

Einstein’s Equation

Use ‘light-cone’ coordinates: $t = \zeta - \eta \qquad z = \zeta + \eta$

Einstein’s vacuum equation:

$$R_{\mu\nu} = 0$$

$$R_{ab} = 0$$

$$R_{00} + R_{33} = 0$$
$$R_{03} = 0$$

$$\underbrace{\left(\alpha g_{,\zeta} g^{-1}\right)}_{-A}{}_{,\eta} + \underbrace{\left(\alpha g_{,\eta} g^{-1}\right)}_{B}{}_{,\zeta} = 0$$

$$\begin{aligned} (\ln f)_{,\zeta} &= \frac{(\ln \alpha)_{,\zeta\zeta}}{(\ln \alpha)_{,\zeta}} + \frac{1}{4\alpha\alpha_{,\zeta}} Tr\left(A^2\right) \\ (\ln f)_{,\eta} &= \frac{(\ln \alpha)_{,\eta\eta}}{(\ln \alpha)_{,\eta}} + \frac{1}{4\alpha\alpha_{,\eta}} Tr\left(B^2\right) \end{aligned}$$

$$\det(g) = \alpha^2$$

$$\alpha_{,\zeta\eta} = 0 \implies \alpha(\zeta, \eta) = a(\zeta) + b(\eta) \qquad \beta(\zeta, \eta) = a(\zeta) - b(\eta)$$

The real challenge is to solve the equation for $g(\zeta, \eta)$

The Lax Pair

$$\underbrace{\left(\alpha g_{,\zeta} g^{-1}\right)}_{-A}{}_{,\eta} + \underbrace{\left(\alpha g_{,\eta} g^{-1}\right)}_{B}{}_{,\zeta} = 0$$

(1) $A_{,\eta} = B_{,\zeta}$

(2) $A_{,\eta} + B_{,\zeta} + \alpha^{-1}[A, B] - \alpha_{,\eta}\alpha^{-1}A - \alpha_{,\zeta}\alpha^{-1}B = 0$

Goal: Express (1) & (2) as the compatibility condition for two linear equations for an unknown matrix $\psi(\lambda, \zeta, \eta)$

$$\begin{aligned} D_1 &= \partial_\zeta - \frac{2\alpha_{,\zeta}\lambda}{\lambda - \alpha}\partial_\lambda & \qquad D_1\psi &= \frac{A}{\lambda - \alpha}\psi \\ D_2 &= \partial_\eta + \frac{2\alpha_{,\eta}\lambda}{\lambda + \alpha}\partial_\lambda & \qquad D_2\psi &= \frac{B}{\lambda + \alpha}\psi \end{aligned} \quad \Longrightarrow \quad \text{Eqs. (1) \& (2)}$$

Taking the limit $\lambda \to 0$ reveals that: $g(\zeta, \eta) = \psi(0, \zeta, \eta)$

The Dressing Method

$$\begin{aligned} D_1\psi &= \frac{A}{\lambda-\alpha}\psi \\ D_2\psi &= \frac{B}{\lambda+\alpha}\psi \end{aligned} \quad \Longrightarrow \quad \begin{aligned} D_1\chi &= \frac{1}{\lambda-\alpha}\left(A\chi-\chi A_0\right) \\ D_2\chi &= \frac{1}{\lambda+\alpha}\left(B\chi-\chi B_0\right) \end{aligned}$$

<u>The problem:</u> Solving for ψ without knowing g

Assume a particular solution $g_0\left(\zeta,\eta\right) \mapsto \psi_0$

Ansatz: $\psi=\chi\psi_0$ χ : the ‘dressing’ matrix (2x2)

The solution is then given by: $g=\chi\left(0\right)g_0$

For physical reasons, g must be real and symmetric.

<u>real</u>

$$\begin{aligned} \bar{\chi}\left(\bar{\lambda}\right) &= \chi\left(\lambda\right) \\ \bar{\psi}\left(\bar{\lambda}\right) &= \psi\left(\lambda\right) \end{aligned}$$

<u>symmetric</u>

$$g=\chi\left(\boxed{\alpha^2/\lambda}\right)g_0\chi^T\left(\lambda\right)$$

$$\Longrightarrow \chi(\infty)=I$$

Gravitational Solitons

We seek a dressing matrix of the form:

$$\chi = I + \sum_{k=1}^{n} \left(\frac{\boxed{R_k}}{\lambda - \boxed{\mu_k}} + \frac{\bar{R}_k}{\lambda - \bar{\mu}_k} \right)$$

$$\begin{aligned} D_1\chi &= \frac{1}{\lambda - \alpha}\left(A\chi - \chi A_0\right) & & & \mu_{k,\zeta} &= \frac{2\alpha_{,\zeta}\mu_k}{\alpha - \mu_k} \\ & & \Longrightarrow & & & \\ \underbrace{D_2\chi} &= \frac{1}{\lambda + \alpha}\left(B\chi - \chi B_0\right) & & & \mu_{k,\eta} &= \frac{2\alpha_{,\eta}\mu_k}{\alpha + \mu_k} \end{aligned}$$

second order pole at $\lambda = \mu_k$

$$\mu_k^2 + 2\left(\beta - \omega_k\right)\mu_k + \alpha^2 = 0$$

$$\mu_k = \omega_k - \beta \pm \sqrt{\left(\omega_k - \beta\right)^2 - \alpha^2}$$

The second root is the pole α^2/μ_k of χ^{-1}

Picture

Gravitational Solitons

$$\chi = I + \sum_{k=1}^{n} \left(\frac{R_k}{\lambda - \mu_k} + \frac{\bar{R}_k}{\lambda - \bar{\mu}_k} \right) \qquad R_k \chi^{-1} (\mu_k) = 0$$

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)} \qquad \left[\chi^{-1} (\mu_k)\right]_{ab} = q_a^{(k)} p_b^{(k)}$$

Determined from the symmetry condition

$$\begin{aligned} \frac{A}{\lambda - \alpha} &= (D_1 \chi) \chi^{-1} + \chi \frac{A_0}{\lambda - \alpha} \chi^{-1} \\ \underbrace{\frac{B}{\lambda + \alpha}}_{} &= (D_2 \chi) \chi^{-1} + \chi \frac{B_0}{\lambda + \alpha} \chi^{-1} \end{aligned}$$

analytic at $\lambda = \mu_k$

$$\left. \begin{aligned} \left(m_{a,\zeta}^{(k)} + m_b^{(k)} \frac{(A_0)_{ba}}{\mu_k - \alpha} \right) q_a^{(k)} &= 0 \\ \left(m_{a,\eta}^{(k)} + m_b^{(k)} \frac{(B_0)_{ba}}{\mu_k + \alpha} \right) q_a^{(k)} &= 0 \end{aligned} \right\}$$

can be solved in general in terms of ψ_0

Summary of the method

1

Take a particular solution $g_0\left(\zeta,\eta\right)$

Can skip this step when g_0 is diagonal

2

$$\overbrace{\begin{aligned} D_1\psi_0 &= \frac{A_0}{\lambda-\alpha}\psi_0 \\ D_2\psi_0 &= \frac{B_0}{\lambda+\alpha}\psi_0 \end{aligned}}$$

$\Longrightarrow$ Get ψ_0

3

$$\chi = I + \sum_{k=1}^{n}\left(\frac{R_k}{\lambda-\mu_k}+\frac{\bar{R}_k}{\lambda-\bar{\mu}_k}\right)$$

$$\mu_k = \omega_k - \beta \pm \sqrt{\left(\omega_k-\beta\right)^2-\alpha^2}$$

$$\left(R_k\right)_{ab} = n_a^{(k)} m_b^{(k)}$$

4

Get an n-solitonic solution!

$$g = \chi\left(0\right)g_0$$

Diagonal Metrics

$$g_0 = \mathrm{diag}(\alpha e^{2s}, \alpha e^{-2s})$$

$$m_1^{(k)} = \frac{A_k}{\sqrt{\mu_k}} \exp\left[-(\rho_k/2 + s)\right] \qquad \rho_{k,\zeta} = \frac{\alpha + \mu_k}{\alpha - \mu_k} 2s_{,\zeta}$$

$$m_2^{(k)} = \frac{A_k}{\sqrt{\mu_k}} \exp\left[(\rho_k/2 + s)\right] \qquad \rho_{k,\eta} = \frac{\alpha - \mu_k}{\alpha + \mu_k} 2s_{,\eta}$$

$\Longrightarrow$ n-soliton solution can be written explicitly in this case, but will depend on the functions ρ_k

E.g. general one-soliton solution for a diagonal background:

$$g = \frac{1}{2\mu \cosh \rho} \begin{pmatrix} (\mu^2 e^{\rho} + \alpha^2 e^{-\rho}) e^{2s} & \alpha^2 - \mu^2 \\ \alpha^2 - \mu^2 & (\alpha^2 e^{\rho} + \mu^2 e^{-\rho}) e^{-2s} \end{pmatrix}$$

One-Soliton on Einstein-Rosen Background

Einstein-Rosen metric:

$$
\begin{aligned}
t &\mapsto r \\
z &\mapsto t \\
x &\mapsto i\theta \\
y &\mapsto iz
\end{aligned}
$$

$$
\begin{aligned}
g_0 &= \mathrm{diag}(\alpha e^{2s}, \alpha e^{-2s}) \\
\alpha(r,t) &= r \\
s(r,t) &= J_0(r)\sin(t)
\end{aligned}
$$

$$\varepsilon_k = \frac{r}{t-\omega_k}$$

$$
\begin{aligned}
\rho_k^{(0)}(r,t) &= 2J_0(r)\sin(t) \\
\rho_k^{(1)}(r,t) &= -2J_1(r)\cos(t) \\
&\vdots
\end{aligned}
$$

In this case ρ_k cannot be integrated analytically.

$$
\begin{aligned}
\mu_k &= \omega_k - \beta \pm \sqrt{(\omega_k-\beta)^2-\alpha^2} \\
&= (\beta-\omega_k)(-1\pm\sqrt{1-\varepsilon_k^2})
\end{aligned}
$$

$$\varepsilon_k = \frac{\alpha}{\beta-\omega_k}$$

Write: $\rho_k = \rho_k^{(0)} + \rho_k^{(1)}\varepsilon_k + \rho_k^{(2)}\varepsilon_k^2 + \ldots$

$\Longrightarrow$ Get a recursive relation for ρ_k in terms of s only!

Pretty Soliton

$$g = \frac{1}{2\mu\cosh\rho}\begin{pmatrix} (\mu^2 e^{\rho} + \alpha^2 e^{-\rho})e^{2s} & \alpha^2 - \mu^2 \\ \alpha^2 - \mu^2 & (\alpha^2 e^{\rho} + \mu^2 e^{-\rho})e^{-2s} \end{pmatrix}$$

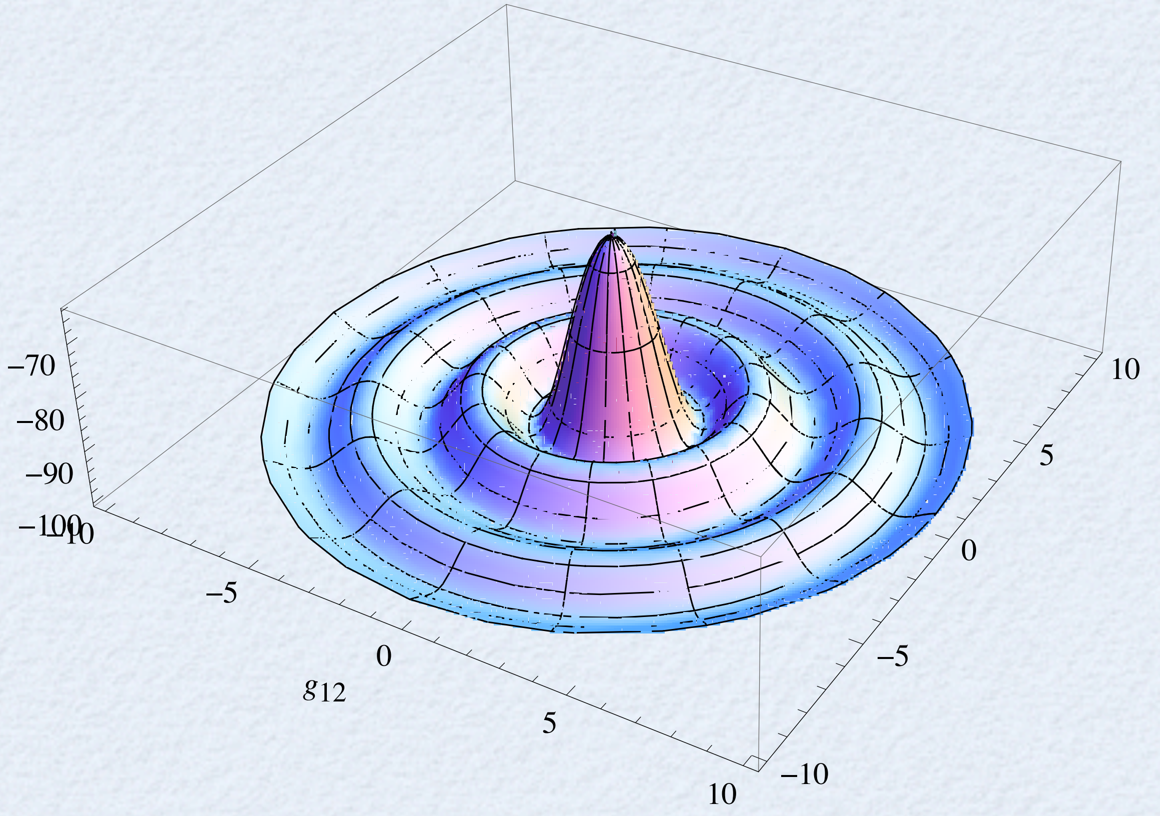
What happens in the limit $\varepsilon \to 1$?

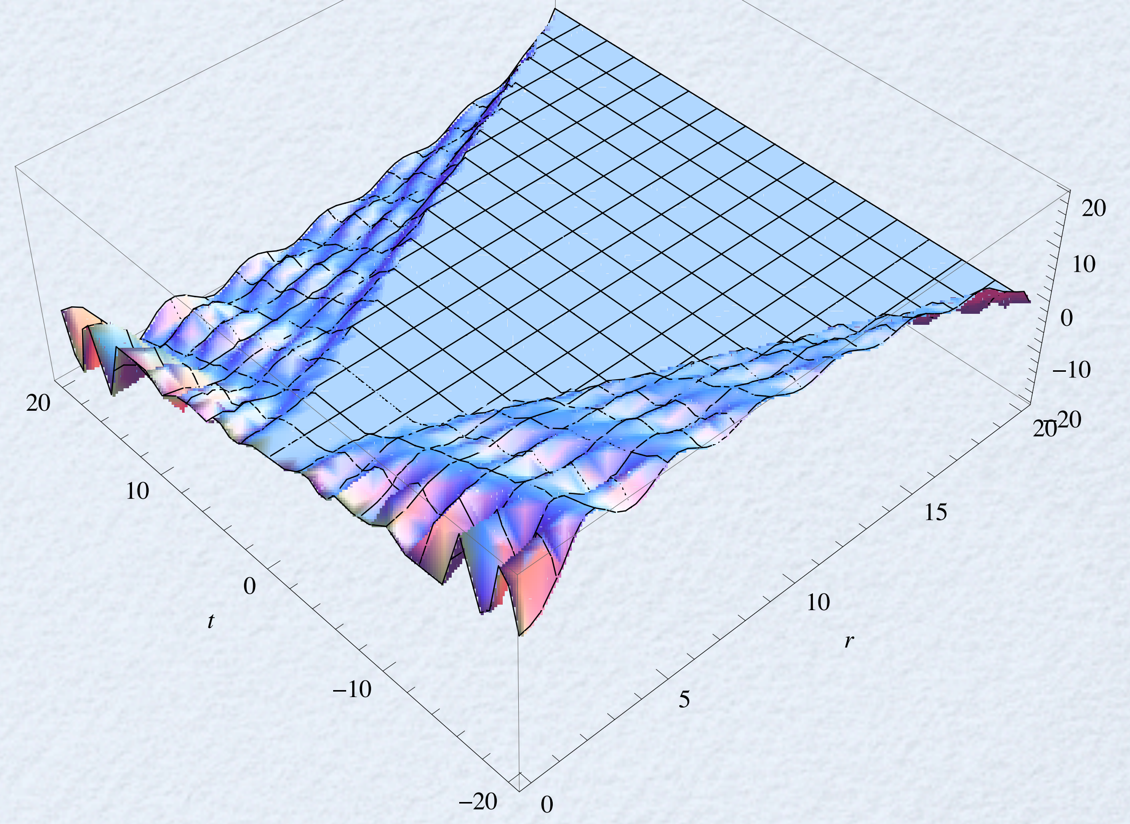
$$\omega = 0$$

$$g \to g_0$$

$$0 \le r \le 10 \qquad t = 100$$

$\Longrightarrow$ Solution can be extended.





Future Goals

- Find infinitely many conservation laws of the Einstein equation (so far I have 5...)
- Study other seeds
- Establish an IST method for metrics of other forms.

The symmetry condition

When g is symmetric, the matrix:

$$\chi'(\lambda) = g\left(\left(\chi\left(\alpha^2/\lambda\right)\right)^{-1}\right)^T g_0^{-1}$$

satisfies the Dressing equations.

Therefore we require

$$\chi'(\lambda) = \chi(\lambda)$$

Or equivalently,

$$g = \chi\left(\alpha^2/\lambda\right) g_0 \chi^T(\lambda)$$

to guarantee the symmetry of g

The residue matrices

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)}$$

$$m_a^{(k)} = m_{0b}^{(k)} M_{ba}^{(k)} \qquad M^{(k)} = \left(\psi_0^{-1}\right)_{\lambda=\mu_k}$$

$$\begin{aligned} \left(m_{a,\zeta}^{(k)} + m_b^{(k)} \frac{(A_0)_{ba}}{\mu_k - \alpha}\right) q_a^{(k)} &= 0 \qquad & M_\zeta^{(k)} + M^{(k)} \frac{A_0}{\mu_k - \alpha} &= 0 \\ \left(m_{a,\eta}^{(k)} + m_b^{(k)} \frac{(B_0)_{ba}}{\mu_k + \alpha}\right) q_a^{(k)} &= 0 \qquad & M_\eta^{(k)} + M^{(k)} \frac{B_0}{\mu_k + \alpha} &= 0 \end{aligned}$$

The symmetry condition: $\underbrace{g}_{\text{}} = \chi\left(\alpha^2/\lambda\right) g_0 \chi^T(\lambda)$

Analytic at $\lambda = \alpha^2/\mu_k$

$$\Longrightarrow \quad R_k g_0 \left[I + \sum_{l=1}^{n} \left(\frac{R_l^T}{\nu_k - \mu_l} + \frac{\bar{R}_l^T}{\nu_k - \bar{\mu}_l} \right) \right] = 0$$

$\Longrightarrow$ Get n algebraic equations for the vectors $n^{(k)}$

The Riemann-Hilbert Representation

Define: $\omega = \frac{1}{2}\left(\frac{\alpha^2}{\lambda} + 2\beta + \lambda\right)$

For any matrix function $G_0(\omega)$

$$\begin{aligned} D_1\omega &= 0 \\ D_2\omega &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} D_1 G_0(\omega) &= 0 \\ D_2 G_0(\omega) &= 0 \end{aligned}$$

Assume $G_0(\omega)$ is analytic on the circle: $|\lambda|^2 = \alpha^2$

Define: $G(\lambda, \zeta, \eta) = \psi_0 G_0 \psi_0^{-1}$

$$\Longrightarrow \quad \begin{aligned} D_1 G &= \frac{1}{\lambda - \alpha}\left(A_0 G - G A_0\right) \\ D_2 G &= \frac{1}{\lambda + \alpha}\left(B_0 G - G B_0\right) \end{aligned}$$

Find matrices: $\begin{cases} \chi_1 : \text{analytic inside the circle} \\ \chi_2 : \text{analytic outside the circle} \end{cases}$

such that

$\chi_1 = \chi_2 G$

$\chi_2(\infty) = I$

The Riemann-Hilbert Representation

$$\underbrace{\begin{matrix}\left(D_1\chi_1+\frac{1}{\lambda-\alpha}\chi_1A_0\right)\chi_1^{-1}\\ \left(D_2\chi_1+\frac{1}{\lambda+\alpha}\chi_1B_0\right)\chi_1^{-1}\end{matrix}}_{\text{analytic in circle}} \begin{matrix}=\\ =\end{matrix} \underbrace{\begin{matrix}\left(D_1\chi_2+\frac{1}{\boxed{\lambda-\alpha}}\chi_2A_0\right)\chi_2^{-1}\\ \left(D_2\chi_2+\frac{1}{\boxed{\lambda+\alpha}}\chi_2B_0\right)\chi_2^{-1}\end{matrix}}_{\text{analytic out of circle}} \begin{matrix}=\frac{1}{\lambda-\alpha}A\\ =\frac{1}{\lambda+\alpha}B\end{matrix}$$

$g=\chi_2\left(0\right)g_0$

Seek solution of the form:

$$\begin{aligned}\chi_1^{-1}\left(\lambda\right) &= I+\frac{1}{\pi i}\oint_\Gamma\frac{\rho\left(z\right)}{\lambda-z}dz\\ \chi_2^{-1}\left(\lambda\right) &= I+\frac{1}{\pi i}\oint_\Gamma\frac{\rho\left(z\right)}{\lambda-z}dz\end{aligned}$$

Sokhotskiy-Plemelj formula $\Longrightarrow$

$$\rho\left(z\right)+C\left(z,\zeta,\eta\right)\left[I+\frac{1}{\pi i}p.v.\oint_\Gamma\frac{\rho\left(z'\right)}{z-z'}dz'\right]=0$$ on the circle

$$C=\left(I-G\right)\left(I+G\right)^{-1}$$